THE HURWITZ-COURANT FUNKTIONENTHEORIE

Vorlesungen über allgemeine Funktionentheorie und elliptische Funktionen. Von Adolf Hurwitz. Herausgegeben und ergänzt durch einen Abschnitt über *geometrische Funktionentheorie*, von R. Courant. Berlin, Julius Springer, 1922. vi + 399 pp.

The present volume is the third of a series * of mathematical monographs under the editorship of R. Courant, which was inaugurated in 1921 with the first volume of Blaschke's *Vorlesungen über Differentialgeometrie*. The announcements of the series and the initial volume have been such as to lead one to expect thoroughly modern and otherwise valuable treatments of the subjects considered, and it is therefore with disappointment that one examines the present volume, in spite of certain merits which it has.

According to the preface, the first two parts are the lecture notes of Hurwitz, almost unchanged, written from the Weierstrassian standpoint. The third part is by Courant, and is based on the ideas of Riemann. Thus one is prepared for differences, but hardly such striking differences as are found. In fact, so lacking is organic connection, so dissimilar in spirit are the works of the two authors and so different their qualifications, that one can only regard the volume as two books printed as one.

Part I consists of 131 pages, devoted to the general theory of functions, the power series being the basis. Cauchy's integral theorem is introduced and used effectively, somewhat beyond the middle of the treatment—it might have been employed earlier with profit. The ground covered is surprisingly extensive; this is partly due to the fact that a knowledge of the foundations of the theory of functions of a real variable is presupposed, and partly to a rare skill in the arrangement, and a willingness on the part of the author to forego extreme generality in his theorems. The style is lucid and the reviewer noticed no faults in the logic. A sketch of the contents follows: complex numbers and their representations on plane and sphere; power series, their circle of convergence, reckoning with them, their identical vanishing, their continuation, their differentiation and integration; the notion of analytic function; the elementary functions; integration, the theory of residues, Laurent series, and related questions; meromorphic functions, the Mittag-Leffler theorem on the partial fraction development of a meromorphic function, the factorization of transcendental integral functions; the inversion of power series and the Lagrange series.

Part II comprises 112 pages on elliptic functions, also lucidly written, and abounding in valuable results and formulas. The transition from Part I is made naturally, by a consideration of the properties of the general periodic meromorphic function. The salient topics which follow are: the Weierstrass function $\wp(u)$, the invariants connected with it, the ex-

* *Die Grundlehren der mathematischen Wissenschaften in Einzeldarstellungen mit besonderer Berücksichtigung der Anwendungsgebiete.*

pression of the general elliptic function in terms of $\wp(u)$ and its derivative; the function $\sigma(u)$; the theta functions; the functions sn u, cn u, and dn u; the elliptic modular functions; the uniformization of certain algebraic relationships; the elliptic integrals; the Landen transformation.

In marked contrast to the first two parts, Part III (149 pages) aims at a high degree of generality. It gives the impression of being the work of a mind endowed with fine intuitive faculties, but lacking in the self-discipline and critical sense which beget confidence. The progress in geometric function theory of the last two decades, and the need of a systematic exposition of the results, have been such as to lead to the hope for an adequate presentation here. What is found may, indeed, serve as an indication of some of the directions which modern investigations have taken—in fact, a very interesting one. But the proofs offered often leave the reader unconvinced as to their validity and, at times, uncertain as to whether they can be made valid.

The initial paragraphs of Part III are devoted to line integrals, Green's theorem, and the mapping properties of the general analytic function. A stationary flow of an incompressible fluid is defined, according to the author, by a vector field, without restriction. The integral over a closed curve of the normal component of the field is declared to represent the diminution of the quantity of the fluid (Flüssigkeitsmenge) in the region bounded by the curve per unit of time! (One is tempted to ask, first, whether *quantity* means volume, or mass; and, secondly, in what sense the quantity of fluid within the circle $x^2 + y^2 = 1$, whose velocity field is given by $u = x$, $v = y$, diminishes by 2π units per second.) This may serve as an example of the author's interest in the "besondere Berücksichtigung der Anwendungsgebiete" of the announcement of the series. A second treatment of Cauchy's integral theorem, and of the logarithmic function exhibits a lack of correlation with the first part of the book.

The boundary value problem of potential theory for the circle is treated by Poisson's integral and a Cesàro summation. There follows an acceptable discussion of the mapping properties of the elementary functions, of the elliptic integral of the first kind, a study of the process of analytic continuation by reflection, of the functions connected with the triangle with straight or circular segments as sides, the differential equation satisfied by the functions connected with polygons of circular arcs, and Picard's theorem. The theorem of Darboux that if $\zeta = f(z)$ is regular within a region B bounded by a regular curve C, and continuous in the closed region, and if $\zeta = f(z)$ maps C on a curve C' without double points, then this function maps the interior of C on the interior of C' in a one-to-one manner, is left half proved. The statements (pp. 299–300) as to the generalizations of the principle of analytic continuation by reflection are, to say the least, insufficiently grounded.

The concluding chapter constitutes the central, and most interesting, portion of Part III. The fundamental mapping theorem is attacked in the form: the general simply or multiply connected open continuum can always be mapped onto a "Schlitzbereich," i.e., the whole plane bounded by certain straight line segments all lying on a set of parallels.* The proof of the theorem, to which 26 pages are devoted, is, in the nature of things, difficult. It is here attacked by the method of minimizing a Dirichlet integral. While there are a number of places where statements are unsubstantiated, the impression is left that the methods are in general direct and well chosen, and that a revision of the treatment so as to complete a real proof would be a possible and worth while labor. It is rather in the extensions that more serious doubts assail one.

The later sections of the chapter take up abelian integrals and algebraic functions on Riemann surfaces, the general problem of uniformization, and the conformal mapping of multiply connected surfaces on each other.

The reader who wishes an orientation with respect to geometric function theory, and who will not be misled as to validity of proofs, will find Part III interesting, and even stimulating. There are excellent possibilities ahead for a revision. The proof reading has been well done, and the typography and general appearance of the book are models of excellence.

O. D. KELLOGG

CORRECTIONS

BY HAROLD HILTON

In the article on pages 303–308 of the July issue of this BULLETIN (vol. 29, No. 7) the following corrections by the author were received by the editors after the final proofs had been returned to the printers.

On page 304, in line 13, change $(p-q)/q$ to $(p-q)/p$.

On page 304, in line 23, change ϕ to ω.

On page 305, in line 19, change $K\pi\epsilon$ to $K\pi/\epsilon$; and change $e^{-k\pi t/\epsilon}\sin k\pi\epsilon$ to $e^{-K\pi t/\epsilon}\sin(k\pi/\epsilon)$.

On page 305, in line 3 from the bottom, change T to τ.

On page 308, omit entirely lines 10–13.

On page 308, in line 3 from the bottom, change (3), (4) to (8), (9).

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* The open continuum in question is supposed to be of finite connectivity in the proof offered. Later it is asserted that the theorem admits of generalization to the case of infinite connectivity, but we are not told what becomes of the Schlitzbereich.