BOLZANO ON PARADOXES

Paradoxien des Unendlichen. By Bernard Bolzano, mit Anmerkungen versehen von Hans Hahn. Leipzig, Meiner, 1921. 12 + 157 pp.

This is a new edition, with notes by Hans Hahn, of the book first published in 1851, three years after Bolzano's death. It appears to have remained for many years almost unknown; for L. Couturat makes no mention of it in his book *De L'Infini Mathématique* (1896) except to state in a note that he came across it when his own book had already been entirely printed. Stolz includes a consideration of this book in an article (MATHEMATISCHE ANNALEN, 1881, p. 255) giving an estimate of Bolzano's work and influence, and states that several years before Cauchy published his lectures on the calculus, Bolzano had developed the fundamental concepts of the calculus which in many respects agree with those of Cauchy, but which in important respects are more complete. Furthermore Hankel attributes to Bolzano priority over Cauchy of the proper conception of infinite series.

The reviewer does not think it fair to criticize severely from the standpoint of present standards of rigor a book written about seventy-five years ago; but in view of the high estimates put on Bolzano's work, it does seem desirable to mention a few of the errors.

Bolzano is concerned with a discussion of Gergonne's solution of the series

(1) $$a - a + a - a + \cdots.$$

The solution is as follows. Let x be the value of the series; then

$$x = a - a + a - a + \cdots = a - (a - a + a - a + \cdots),$$

that is,

$$x = a - x, \quad \text{and} \quad x = \frac{a}{2}.$$

Bolzano criticizes this solution in two ways. In the first place he says that the series in parentheses is not identical with the series (1), because regarded as a set of terms it lacks the first term a. This assertion of Bolzano's is of course erroneous; in fact if it be granted that the series (1) have a value at all, then Gergonne's solution is correct. In the second place Bolzano objects altogether to attaching a value to the series (1). While this objection is quite legitimate, the *grounds* for the objection are not. The series can have no value, Bolzano says, since if it did have, it would simultaneously be equal to 0, a and $-a$, inasmuch as it can be written in several forms, as follows:

$$\begin{aligned}
a - a + a - a + \cdots \\
= (a - a) + (a - a) + \cdots \\
= a + (- a + a) + (- a + a) + \cdots \\
= (- a + a) + (- a + a) + \cdots \\
= - a + (a - a) + (a - a) + \cdots.
\end{aligned}$$

The author's demand, however, that a satisfactory definition for the value of a series must allow for the interchange of the order of terms and for the insertion and omission of parentheses, is of course an unreasonable one; it is not indeed satisfied even for all convergent series.

Bolzano now discusses the two series

$$S_1 = 1 + 2 + 3 + 4 + \cdots,$$
$$S_2 = 1 + 2^2 + 3^2 + 4^2 + \cdots.$$

It is pointed out, on the one hand, that $S_1 < S_2$, inasmuch as each term of S_1 (except the first) is less than the corresponding term of S_2; but on the other hand, $S_1 > S_2$, since every term of S_2 occurs somewhere in S_1 and there are terms in S_1 which do not occur in S_2. Bolzano's explanation of this paradox is that in reality $S_2 > S_1$; for one has no right to infer that the first of two infinite series has a greater value than the second simply on the ground that the first contains all the terms of the second and many other terms besides. This explanation is of course entirely incorrect. The fact is that neither of the two series can be asserted to be greater in value than the other, because as neither of them possesses a value, their values cannot be compared.

There are misstatements in the theory of functions. The author states (p. 65, footnote) that every continuous function, except possibly for isolated values of the independent variable, possesses a derivative, and (p. 68) can be expanded by Taylor's Theorem.

On the other hand Bolzano must be credited with ideas that were certainly in advance of his time. For example, he gives a proof (p. 13) of the existence of an infinite set—by proving that the number of propositions is infinite—and this anticipates Dedekind by nearly forty years. Again, he points out the possibility of setting up a one-one correspondence between the elements of an infinite set and those of a proper part of that set—an idea that was to become fundamental in the work of Cantor and Dedekind.

The book deals with paradoxes of the infinite. Some of these so-called paradoxes are, from the standpoint of present knowledge, not paradoxes at all. Such, for example, are those involving infinite series, mentioned above. Others are cleared up by an accurate application of the definitions of infinity and infinitesimal.

The book is of interest to the philosopher and to the theologian as well as to the mathematician. Bolzano was an Austrian priest, who as Professor of the Philosophy of Religion at Prague tried in his lectures to reconcile Catholic theology with the ideas of modern science. Twenty-five pages of the book are devoted to purely metaphysical considerations, which include a development of the doctrine of panpsychism and an application of this doctrine to the problem of interaction between mind and matter.

The notes by Hans Hahn constitute a helpful guide to the text. Explanations are given only to the mathematical portions of the text (pp. 1–107); the twenty-five pages of metaphysical discussion are not considered by Hahn.

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