DETERMINATION OF ALL SYSTEMS OF ∞^4 CURVES IN SPACE IN WHICH THE SUM OF THE ANGLES OF EVERY TRIANGLE IS TWO RIGHT ANGLES *

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1. *Introduction.* Consider the curves which intersect an arbitrarily chosen system of ∞^1 curves in the plane under a fixed angle α. If α is varied, a system of ∞^2 curves is obtained, termed an *isogonal* family. Isogonal families are characterized by differential equations of the form

$$y'' = (T_x + y' T_y)(1 + y'^2), \tag{1}$$

where T is any function of x and y.

It is easy to prove synthetically that in all isogonal families, and in no other systems of ∞^2 curves in the plane, the sum of the angles of the triangle formed by any three of the curves is equal to π.†

A *natural* family of curves in any space is one obtainable as the system of extremals of a calculus of variations problem of the form

$$\int F ds = \text{minimum}, \tag{2}$$

where F is any point function.‡ In the plane, F is a function of x and y, and the Euler-Lagrange equation of (2) is

$$y'' = (L_y - y' L_x)(1 + y'^2), \tag{3}$$

where $L = \log F$.

Since the family formed by the ∞^2 straight lines of the plane is both isogonal and natural, and since each of these characters is invariant under conformal transformation, every

* Presented to the Society, April 28, 1923.

† G. Scheffers, *Isogonalkurven, Äquitangentialkurven und komplexe Zahlen*, MATHEMATISCHE ANNALEN, vol. 60 (1905), p. 504.

‡ See E. Kasner, PRINCETON COLLOQUIUM LECTURES (1912), pp. 34–37.

curve family conformally equivalent to the straight lines must be both isogonal and natural.

Conversely, for a curve family which is at once isogonal and natural, we have by identification of (1) with (3)

$$(4) \qquad T_x = L_y, \qquad T_y = -L_x.$$

These equations imply that $L - iT$ and consequently e^{L-iT} are analytic functions of $x + iy$. Since the conformal transformation

$$x_1 + iy_1 = e^{L-iT},$$

transforms $e^L ds$ into ds_1, it converts the extremals of $\int F ds = \int e^L ds =$ minimum into those of $\int ds_1 =$ minimum, that is, into the straight lines.

The conditions (4) can be satisfied only when T is Laplacian. It follows that the property of having the angle sum in each triangle equal to π is, in the plane, not restricted to the curve families derivable by conformal transformation from the straight lines. In contrast with this fact, it is the object of the present paper to prove the following theorem.

THEOREM. *If a system of ∞^4 curves in space is such that in each of its triangles the sum of the angles is equal to π, then it is either the system formed by the ∞^4 straight lines of space, or an image of that system by a conformal transformation of space, namely the ∞^4 circles through a fixed point.*

It is presumed in the statement of this theorem that the only systems of ∞^4 curves in space taken into consideration are those that can be defined by a system of two differential equations of the form

$$(5) \qquad y'' = F(x, y, z, p, q), \qquad z'' = G(x, y, z, p, q),$$

where p, q represent y', z' respectively, and where F and G are analytic functions of their five arguments. We select a region of the x, y, z, p, q continuum, within which each of these functions has a branch which is uniform and regular, and we restrict ourselves in the calculations that follow to this region and to these branches.

2. *A Triangle with One Infinitesimal Angle.* We shall use the symbol $\mathfrak{F}$ to denote any quadruply infinite family of curves defined by equations of the form (5). Under the restrictions just stated, there will pass through each point and in each direction a unique curve of $\mathfrak{F}$.

Choose any point 0—denote its coordinates by x_0, y_0, z_0—and let $1 : p_0 : q_0$ and $1 : p_0 + \delta p_0 : q_0 + \delta q_0$ define two directions through 0 infinitely near to one another. These determine in the family $\mathfrak{F}$ two consecutive curves C_1 and C_2. Let the equations of C_1 be

$$Y = y(X), \qquad Z = z(X); \tag{6}$$

then those of C_2 will be

$$\begin{aligned} Y &= y(X) + \delta p_0 \eta_1(X) + \delta q_0 \eta_2(X), \\ Z &= z(X) + \delta p_0 \zeta_1(X) + \delta q_0 \zeta_2(X), \end{aligned} \tag{7}$$

where η_1, ζ_1 and η_2, ζ_2 obey the equations of variation of the system (5)

$$\begin{aligned} \eta'' &= F_p \eta' + F_q \zeta' + F_y \eta + F_z \zeta, \\ \zeta'' &= G_p \eta' + G_q \zeta' + G_y \eta + G_z \zeta, \end{aligned} \tag{8}$$

and are completely determined by the additional data

$$(9_1) \quad \eta_1(x_0) = 0, \quad \zeta_1(x_0) = 0, \quad \eta_1'(x_0) = 1, \quad \zeta_1'(x_0) = 0,$$

$$(9_2) \quad \eta_2(x_0) = 0, \quad \zeta_2(x_0) = 0, \quad \eta_2'(x_0) = 0, \quad \zeta_2'(x_0) = 1.$$

It is to be understood that in the coefficients of (8), which are originally functions of x, y, z, p, q, we are to substitute by means of (6)

$$x = X, \quad y = y(X), \quad z = z(X), \quad p = y'(X), \quad q = z'(X),$$

so that these coefficients become functions only of X, the abscissa along C_1.

On C_1 let any point $1(x, y, z)$ other than 0 be selected, and let $2(x + \delta x, y + \delta y, z + \delta z)$ be an infinitely near point on C_2. Through 1 and 2 there passes a unique curve of the family $\mathfrak{F}$; designate it as $\bar{C}$, and denote by $1 : \bar{p} : \bar{q}$ and $1 : \bar{p} + \delta\bar{p} : \bar{q} + \delta\bar{q}$

its directions at 1 and 2 respectively. Then

$$\delta y = \bar{p}\delta x, \qquad \delta z = \bar{q}\delta x, \tag{10}$$

and

$$\delta\bar{p} = \overline{F}\delta x, \qquad \delta\bar{q} = \overline{G}\delta x, \tag{11}$$

where the bars over F and G indicate that these functions are to be formed for the arguments $x, y, z, \bar{p}, \bar{q}$.

Allow $1 : p : q$ and $1 : p + \delta p : q + \delta q$ to represent respectively the direction of C_1 at 1 and of C_2 at 2; then by differentiation of (6) and (7), substitution of the coordinates of 1 and 2, and use of (5),

$$p = y'(x), \qquad q = z'(x),$$

and

$$\begin{cases} \delta p = F\delta x + \delta p_0 \eta_1' + \delta q_0 \eta_2', \\ \delta q = G\delta x + \delta p_0 \zeta_1' + \delta q_0 \zeta_2', \end{cases} \tag{12}$$

where the arguments in F, G are the x, y, z, p, q of the point 1 and curve C_1, and in $\eta_1', \zeta_1', \eta_2', \zeta_2'$ the argument is x.

The fact that the curve C_2 or (7) contains the point 2 gives, with the use of (10),

$$(\bar{p} - p)\delta x = \delta p_0 \eta_1 + \delta q_0 \eta_2,$$
$$(\bar{q} - q)\delta x = \delta p_0 \zeta_1 + \delta q_0 \zeta_2;$$

therefore

$$\begin{cases} \delta x : \delta p_0 : \delta q_0 = \eta_1\zeta_2 - \eta_2\zeta_1 : [(\bar{p} - p)\zeta_2 - (\bar{q} - q)\eta_2] \\ \qquad\qquad\qquad : [-(\bar{p} - p)\zeta_1 + (\bar{q} - q)\eta_1]. \end{cases} \tag{13}$$

In the curvilinear triangle 012 or $C_1C_2\overline{C}$ let $\delta\omega$ denote the interior angle at 0, θ the interior angle at 1, and $\theta + \delta\theta$ the exterior angle at 2. Then the condition for an angle sum equal to two right angles is

$$\delta\theta = \delta\omega. \tag{14}$$

Now

$$\cos\theta = \frac{1 + p\bar{p} + q\bar{q}}{\sqrt{1 + p^2 + q^2}\sqrt{1 + \bar{p}^2 + \bar{q}^2}}.$$

Therefore

$$
\begin{aligned}
-\sin\theta\delta\theta &= \frac{1+p\bar{p}+q\bar{q}}{\sqrt{1+p^2+q^2}\sqrt{1+\bar{p}^2+\bar{q}^2}} \\
&\quad\times\left\{\frac{\bar{p}\delta p+\bar{q}\delta q+p\delta\bar{p}+q\delta\bar{q}}{1+p\bar{p}+q\bar{q}}\right. \\
&\qquad\left.-\frac{p\delta p+q\delta q}{1+p^2+q^2}-\frac{\bar{p}\delta\bar{p}+\bar{q}\delta\bar{q}}{1+\bar{p}^2+\bar{q}^2}\right\},
\end{aligned}
\tag{15}
$$

where the value of $\sin\theta$ is

$$
\frac{\sqrt{(1+q^2)(\bar{p}-p)^2-2pq(\bar{p}-p)(\bar{q}-q)+(1+p^2)(\bar{q}-q)^2}}{\sqrt{1+p^2+q^2}\sqrt{1+\bar{p}^2+\bar{q}^2}}.
\tag{16}
$$

Besides,

$$
\delta\omega=\frac{\sqrt{(1+q_0{}^2)\delta p_0{}^2-2p_0q_0\delta p_0\delta q_0+(1+p_0{}^2)\delta q_0{}^2}}{1+p_0{}^2+q_0{}^2}.
\tag{17}
$$

Combining the equations (14) to (17), we have

$$
\begin{aligned}
&-(1+p\bar{p}+q\bar{q})\left\{\frac{\bar{p}\delta p+\bar{q}\delta q+p\delta\bar{p}+q\delta\bar{q}}{1+p\bar{p}+q\bar{q}}\right. \\
&\qquad\left.-\frac{p\delta p+q\delta q}{1+p^2+q^2}-\frac{\bar{p}\delta\bar{p}+\bar{q}\delta\bar{q}}{1+\bar{p}^2+\bar{q}^2}\right\} \\
&=\frac{1}{1+p_0{}^2+q_0{}^2}\sqrt{(1+q_0{}^2)\delta p_0{}^2-2p_0q_0\delta p_0\delta q_0+(1+p_0{}^2)\delta q_0{}^2} \\
&\times\sqrt{(1+q^2)(\bar{p}-p)^2-2pq(\bar{p}-p)(\bar{q}-q)+(1+p^2)(\bar{q}-q)^2}.
\end{aligned}
\tag{18}
$$

By means of the substitutions indicated by (11) and (12), the use of (13), the introduction of

$$
\left\{
\begin{aligned}
\phi &= \frac{(1+q^2)F-pqG}{1+p^2+q^2}, \\
\psi &= \frac{-pqF+(1+p^2)G}{1+p^2+q^2},
\end{aligned}
\right.
\tag{19}
$$

also of symbols ω_i to represent the two-rowed determinants in the matrix

$$
\begin{vmatrix} \eta_1 & \zeta_1 & \eta_1' & \zeta_1' \\ \eta_2 & \zeta_2 & \eta_2' & \zeta_2' \end{vmatrix}
$$

as follows,

$$\omega_1 = (\eta\zeta), \qquad \omega_2 = (\eta\eta'), \qquad \omega_3 = (\eta\zeta'),$$
$$\omega_4 = (\zeta\eta'), \qquad \omega_5 = (\zeta'\zeta), \qquad \omega_6 = (\eta'\zeta'),$$

and of

$$\begin{aligned} \Omega_1 &= (1+q^2)\omega_4 + pq\omega_5, \\ \Omega_2 &= -(1+q^2)\omega_2 + pq\omega_3 - pq\omega_4 - (1+p^2)\omega_5, \\ \Omega_3 &= pq\omega_2 - (1+p^2)\omega_3, \end{aligned}$$

finally of

$$(20)\quad \begin{cases} I = (1+p_0^2)\eta_1^2 + 2p_0q_0\eta_1\eta_2 + (1+q_0^2)\eta_2^2, \\ II = (1+p_0^2)\eta_1\zeta_1 + p_0q_0(\eta_1\zeta_2+\eta_2\zeta_1) + (1+q_0^2)\eta_2\zeta_2, \\ III = (1+p_0^2)\zeta_1^2 + 2p_0q_0\zeta_1\zeta_2 + (1+q_0^2)\zeta_2^2, \end{cases}$$

the condition (18) reduces to

$$\begin{aligned} &(1+p^2+q^2)\omega_1\{(\bar\phi-\phi)(\bar p-p)+(\bar\psi-\psi)(\bar q-q)\} \\ &+\Omega_1(\bar p-p)^2+\Omega_2(\bar p-p)(\bar q-q)+\Omega_3(\bar q-q)^2 \\ (21)\quad &= \frac{1+p^2+q^2}{1+p_0^2+q_0^2}\sqrt{III(\bar p-p)^2-2II(\bar p-p)(\bar q-q)+I(\bar q-q)^2} \\ &\times\sqrt{(1+q^2)(\bar p-p)^2-2pq(\bar p-p)(\bar q-q)+(1+p^2)(\bar q-q)^2}. \end{aligned}$$

It is to be observed that ω_1, Ω_1, Ω_2, Ω_3, and I, II, III are independent of $\bar p$, $\bar q$.

Since the right member of (21) and the second line of the left member are homogeneous functions of the second degree in $\bar p - p$, $\bar q - q$, the same must be true of

$$(22)\qquad (\bar\phi-\phi)(\bar p-p)+(\bar\psi-\psi)(\bar q-q).$$

By Taylor's theorem,

$$\begin{aligned} \bar\phi-\phi &= \phi_p(\bar p-p)+\phi_q(\bar q-q) \\ &\quad+\tfrac{1}{2}\{\phi_{pp}(\bar p-p)^2+2\phi_{pq}(\bar p-p)(\bar q-q)+\phi_{qq}(\bar q-q)^2\}+\cdots, \\ \bar\psi-\psi &= \psi_p(\bar p-p)+\psi_q(\bar q-q) \\ &\quad+\tfrac{1}{2}\{\psi_{pp}(\bar p-p)^2+2\psi_{pq}(\bar p-p)(\bar q-q)+\psi_{qq}(\bar q-q)^2\}+\cdots. \end{aligned}$$

Necessary conditions that (22) be homogeneous of the second degree in $\bar{p} - p$, $\bar{q} - q$ are seen to be

$$\phi_{pp} = 0, \quad 2\,\phi_{pq} + \psi_{pp} = 0, \quad \phi_{qq} + 2\,\psi_{pq} = 0, \quad \psi_{qq} = 0,$$

a system of partial differential equations whose solution is

$$\begin{aligned} \phi &= \beta q^2 - \gamma pq + \lambda p + \mu q + \nu, \\ \psi &= \gamma p^2 - \beta pq + \lambda' p + \mu' q + \nu'; \end{aligned} \tag{23}$$

and these forms of ϕ, ψ are seen to be also sufficient for the condition in question. β, γ, λ, etc. are functions only of x, y, z.

The left member of (21) now becomes a rational entire function of $\bar{p} - p$, $\bar{q} - q$; in order that the same be true of the right member we must have

$$\frac{I}{1 + p^2} = \frac{II}{pq} = \frac{III}{1 + q^2}\,. \tag{24}$$

By means of (8), (9), and (20), I, II, III can be expressed as power series in $t = x - x_0$; we find

$$\begin{aligned} I &= (1 + p_0^2)t^2 + \{(1 + p_0^2)F_p{}^0 + p_0q_0F_q{}^0\}t^3 + \cdots, \\ II &= p_0q_0t^2 + \{(1 + p_0^2)G_p{}^0 + p_0q_0(F_p{}^0 + G_q{}^0) \\ &\qquad + (1 + q_0^2)F_q{}^0\}t^3 + \cdots, \\ III &= (1 + q_0^2)t^2 + \{p_0q_0G_p{}^0 + (1 + q_0^2)G_q{}^0\}t^3 + \cdots. \end{aligned} \tag{25}$$

Furthermore,

$$\begin{aligned} 1 + p^2 &= 1 + p_0^2 + 2p_0F_0t + \cdots, \\ pq &= p_0q_0 + (p_0G_0 + q_0F_0)t + \cdots, \\ 1 + q^2 &= 1 + q_0^2 + 2q_0G_0t + \cdots. \end{aligned} \tag{26}$$

When, after dividing the members of (25) by the corresponding ones of (26), we equate coefficients of t^3, we obtain

$$\begin{aligned} &\frac{(1 + p^2)F_p + pqF_q - 2pF}{1 + p^2} \\ &= \frac{(1+p^2)G_p+pq(F_p+G_q)+(1+q^2)F_q-(pG+qF)}{2pq} \\ &= \frac{pqG_p + (1 + q^2)G_q - 2qG}{1 + q^2}, \end{aligned} \tag{27}$$

where the subscript and superscript 0 has been dropped with evident justification.

Writing

$$
(28) \qquad \begin{aligned} F &= (1 + p^2)\phi + pq\psi, \\ G &= pq\phi + (1 + q^2)\psi, \end{aligned}
$$

obtained by reversion of (19), and then applying the values of ϕ and ψ given by (23), we find that (27) imposes the conditions

$$
\nu = \beta, \qquad \nu' = \gamma, \qquad \lambda' = 0, \qquad \mu = 0, \qquad \lambda = \mu',
$$

which by (23) and (28) reduce F and G to

$$
(29) \qquad \begin{aligned} F &= (\beta - \alpha p)(1 + p^2 + q^2), \\ G &= (\gamma - \alpha q)(1 + p^2 + q^2), \end{aligned}
$$

where we have replaced $\lambda = \mu'$ by $-\alpha$.

3. *The* ∞^3 *Surfaces* Σ. Relative to the ∞^4 straight lines of space there is a family of ∞^3 surfaces, namely the planes, of which the straight lines are the mutual intersections. Not every system of ∞^4 curves in space has a so related family of ∞^3 surfaces. We have found the necessary and sufficient conditions to be

$$
(30) \qquad \begin{aligned} &F_q(F_p + G_q) + 4F_z - 2F_q' = 0, \\ &F_p^2 - G_q^2 + 4(F_y - G_z) - 2(F_p' - G_q') = 0, \\ &G_p(F_p + G_q) + 4G_y - 2G_p' = 0. \end{aligned}
$$

The accent denotes the operator

$$
\frac{\partial}{\partial x} + p\frac{\partial}{\partial y} + q\frac{\partial}{\partial z} + F\frac{\partial}{\partial p} + G\frac{\partial}{\partial q} \cdot
$$

Let A be any point of space, through which d_1 and d_2 are any two directions, and let Γ_1 and Γ_2 denote the curves of the family $\mathfrak{F}$ which pass through A in these directions respectively. Choose arbitrarily a point B on Γ_1 and a point C on Γ_2. There is a unique curve BC of $\mathfrak{F}$ which passes through B and C.

On BC let D be any point. Suppose the curve determined by A and D to have at A the direction d.

Then under the hypothesis of an angle sum in every triangle equal to π, the direction d must belong to the flat pencil determined by d_1 and d_2. For by applying this hypothesis to the triangles ABD, ACD, ABC, it may be deduced that

$$\measuredangle d_1 d + \measuredangle d d_2 = \measuredangle d_1 d_2,$$

which implies that d is coplanar with d_1 and d_2.

The curves of $\mathfrak{F}$ which radiate from A in the directions of the pencil determined by d_1 and d_2 form a surface Σ. By the above, Σ contains AD; therefore it contains D; therefore it contains the curve BC, since D was an arbitrary point of BC.

Now B and C were arbitrary points on Γ_1 and Γ_2 respectively. It follows that through each pair of intersecting curves of $\mathfrak{F}$ there passes a surface Σ which carries ∞^2 curves of $\mathfrak{F}$. It is easy to see how this implies that the ∞^4 curves of $\mathfrak{F}$ are the mutual intersections of ∞^3 surfaces.

The functions F and G, so far reduced to the forms (29), must therefore obey (30). The imposition of (30) restricts F and G further, namely to the forms

$$\left\{\begin{array}{l} F = (L_y - pL_x)(1 + p^2 + q^2), \\ G = (L_z - qL_x)(1 + p^2 + q^2), \end{array}\right. \tag{31}$$

where either

$$L = f(x^2 + y^2 + z^2), \tag{32a}$$

or

$$L = g(y). \tag{32b}$$

In the reduction of L, use is made, in general, of a transformation of the axes.

In other words, the differential equations of $\mathfrak{F}$ are now reduced to one or the other of the forms

$$\begin{array}{ll} y'' = \rho(y - y'x)(1 + y'^2 + z'^2), & \\ z'' = \rho(z - z'x)(1 + y'^2 + z'^2), & \rho = 2f'(x^2 + y^2 + z^2); \end{array} \tag{33a}$$

$$\begin{array}{l} y'' = g'(y)(1 + y'^2 + z'^2), \\ z'' = 0. \end{array} \tag{33b}$$

4. *Final Conditions.* We consider first (33a). Its two equations can be combined so as to give

$$\begin{vmatrix} x & y & z \\ 1 & y' & z' \\ 0 & y'' & z'' \end{vmatrix} = 0,$$

whose general integral is $Ax + By + Cz = 0$, where A, B, C are arbitrary constants. It follows that each curve of $\mathfrak{F}$ lies in a plane that passes through the origin O. The ∞^2 curves of $\mathfrak{F}$ carried by each surface Σ are therefore the sections of Σ by the planes through O (provided that Σ is not itself a plane through O *).

We next observe that the equations (31) are the Euler-Lagrange equations for $\int e^L ds =$ minimum. But if a curve on a surface Σ is an extremal of $\int e^L ds$ relative to space, it is a fortiori an extremal relative to Σ. Thus the ∞^2 curves on Σ form, according to the definition in § 1, a *natural* family on Σ.

Imagine Σ to be represented conformally on the plane. Then this natural family on Σ goes over into a natural family in the plane. For conformal transformation multiplies ds by a point function; therefore the extremals of an integral of the form $\int$ *point function*$\cdot ds$ are transformed into the extremals of an integral of the same form.

Moreover, in this family of ∞^2 curves in the plane the sum of the angles of every triangle is equal to π; the family is therefore *isogonal* as well as natural. By § 1, it is therefore convertible into the ∞^2 straight lines by a conformal transformation of the plane. In this way, it is possible to convert the original system of ∞^2 curves on Σ into the straight lines of the plane by a conformal representation of Σ on the plane.

We have therefore to deal with the following problem: If O is a point of space, and Σ a surface not a plane through O, what is implied by the circumstance that Σ admits of a

* Each curve of $\mathfrak{F}$ belongs to ∞^1 surfaces Σ. These cannot all be planes through O unless the curve is a straight line through O, a case which may be laid aside without affecting the validity of our ultimate conclusion.

conformal representation on the plane in which its ∞^2 sections by the planes through O go over into the ∞^2 straight lines of the plane?

We say that Σ must be either a plane, or else a sphere passing through O.

For consider any minimal line of the plane. In the conformal representation it must correspond to a curve on Σ which (1) is a plane curve, (2) has zero length. A plane curve of zero length is necessarily a minimal straight line. The surface Σ therefore carries two systems of minimal straight lines.

A surface doubly covered by minimal straight lines is either a plane or a sphere. If Σ is a plane, the condition in question is satisfied without further restriction.

But if Σ is a sphere, it must, in addition, pass through O. For the angle sum in a triangle formed by three circles on a sphere, whose planes intersect in a point O, is greater than π if O is interior to the sphere, and less than π if O is exterior to the sphere. Moreover, the condition Σ is a sphere which passes through O, is a sufficient one, for then the stereographic projection of Σ from O as pole is a conformal representation of Σ on the plane which converts the sections of Σ by the planes through O into straight lines.

It follows that each curve of $\mathfrak{F}$ must be either a straight line, or a circle through O. Since each of the three curve families $\mathfrak{F}$, the straight lines of space, the circles through O, is an irreducible analytic manifold of four dimensions, $\mathfrak{F}$ must be identical either with the ∞^4 straight lines of space, or with the ∞^4 circles through O.

The case (33*b*) can by a similar argument be proved to lead only to the straight lines.

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