NOTE ON A CERTAIN TYPE OF RULED SURFACE *

BY W. C. GRAUSTEIN

In the January, 1923, number of this BULLETIN, J. K. Whittemore discusses ruled surfaces having the property that any two secondary asymptotic lines cut equal segments from the rulings. It so happens that, in investigating the determination of a surface by the linear element of its spherical representation and its total and mean curvatures, the present writer was led to consider the same class of ruled surfaces, with the results which are set forth in this note. The method of attack differs from that of Whittemore and the facts obtained overlap only in the case of the characteristic property, namely, that the rulings are parallel to a plane and the parameter of distribution is constant.

Any two secondary asymptotic lines of a ruled surface cut equal segments from the rulings if and only if the surface, when referred to its asymptotic lines as the parametric curves, admits a representation of the usual form,

$$x_i = \xi_i(v) + u\eta_i(v), \qquad (i = 1, 2, 3), \tag{1}$$

where η_1, η_2, η_3 are the direction cosines of the ruling, η, and u is the algebraic distance along the ruling from the directrix, $\xi = \xi(v)$. Analytically, this condition amounts to demanding that $D'' = 0$. But HD'' is a quadratic polynomial in u, whose coefficients are functions of v alone. Thus the condition $D'' = 0$ gives rise to three equations, namely:

$$(\eta\xi'\xi'') = 0, \qquad (\eta\eta'\xi'') + (\eta\xi'\eta'') = 0, \qquad (\eta\eta'\eta'') = 0. \tag{2}$$

The vanishing of the last determinant is the condition that the director cone degenerate into a plane. The spherical indicatrix, $\eta_i = \eta_i(v)$, $(i = 1, 2, 3)$, is then a circle of unit radius, in particular, the circle in which the director plane cuts the unit sphere. If we choose as the parameter v the arc of this circle, it follows that $\eta_i'' = -\eta_i$, $(i = 1, 2, 3)$, and conditions (2) become

$$(\eta\xi'\xi'') = 0, \qquad (\eta\eta'\xi'') = 0. \tag{3}$$

---

* Presented to the Society, April 28, 1923.

The first of these conditions requires merely that the directrix be a secondary asymptotic line and can always be assumed fulfilled. The second is the condition that the parameter of distribution be constant, for the value of this parameter is readily shown to be $\pm\ (\eta\eta'\xi')$. The characteristic property is thus established.

If we exclude the trivial case of a developable, when the surface is necessarily a plane, the directions η, ξ', η' are distinct and not parallel to a plane. Consequently, equations (3) are satisfied only if η and ξ'' are linearly dependent, that is, only if *

$$\eta_i = \frac{\xi_i''}{\sqrt{\xi''|\xi''}}, \qquad (i = 1, 2, 3), \tag{4}$$

where there is no loss of generality in admitting merely the positive square root. Moreover, the assumption $(\xi''|\xi'') \neq 0$ is readily justified, since not all the secondary asymptotic lines can be straight and one which is not can be chosen as the directrix. If we introduce the curve y defined by the equations $y_i = \xi_i'$, $(i = 1, 2, 3)$, the representation (1) of the surface becomes

$$x_i = \int y_i dv + u\eta_i, \qquad (i = 1, 2, 3). \tag{5}$$

Furthermore, we have, in place of (4),

$$\eta_i = \frac{y_i'}{\sqrt{y'|y'}}, \qquad (i = 1, 2, 3).$$

In other words, the circle $\eta = \eta(v)$, which is the spherical indicatrix of the surface, is also the tangent indicatrix of the curve y. This curve, then, is a plane curve. Its plane is parallel to the director plane, but not coincident with it, since otherwise (5) would represent a developable (a plane).

If $y = y(v)$ is an arbitrary plane curve not in a plane through the origin and v is the arc of its tangent indicatrix, $\eta = \eta(v)$, the general ruled surface having the property that the secondary asymptotic lines cut equal segments from the rulings is represented by equations (5).

* If a denotes the number triple a_1, a_2, a_3, then $(a|a) = a_1^2 + a_2^2 + a_3^2$.

We next investigate under what conditions the surface is a conoid. The asymptotic line $u = u_0$ is straight only when the vector product of x_v and x_{vv} vanishes for $u = u_0$. But this vector product is equal to the scalar, $R - u$, multiplied by a non-vanishing triple, where R is the radius of curvature of the curve y. Hence it is zero for some constant value of u only if R is constant.

The surface is a conoid if and only if the curve y is a circle.

For $u = R = \text{const.}$, $x_{vv} = 0$. Hence the arc of the directrix line of the conoid can be taken proportional to v, or, since the arc v of the unit circle, $\eta = \eta(v)$, is also the corresponding angle at the center of the circle, the distance between two arbitrary points of the directrix line is proportional to the angle between the rulings through these points. In fact, if we introduce $u - R = \bar{u}$ as a new parameter, equations (5) become

$$x_i = a_i v + \bar{u}\eta_i(v), \qquad (i = 1, 2, 3), \tag{6}$$

where a_1, a_2, a_3 are constants defining a direction not parallel to the director plane.

The secondary asymptotic lines of a conoid cut equal segments from the rulings if and only if the angle through which a variable ruling turns is proportional to the distance along the directrix line through which the ruling slides.

A right conoid with this property is necessarily a right helicoid, the only minimal ruled surface. It is in this case alone that the curve y of (5) is a circle subtending at the origin a cone of revolution.

If the (x_1, x_2)-plane be taken as the director plane, η_1, η_2, η_3 can be chosen respectively as $\cos v$, $\sin v$, 0, and equations (5) become

$$x_1 = \int dv \int R \cos v \, dv,$$

$$x_2 = \int dv \int R \sin v \, dv,$$

$$x_3 = cv,$$

where R is an arbitrary function of v and c an arbitrary constant, not zero. Whittemore's equations (1″), taken in conjunction with his relation (6), are reducible to this form.

HARVARD UNIVERSITY

THE DIFFERENTIATION OF A FUNCTION OF A FUNCTION

BY H. S. CARSLAW

The reviewer of Rothe's interesting work *Vorlesungen über Höhere Mathematik* (1921) in this BULLETIN (vol. 28, p. 468) calls attention to the author's tentative claim that the first valid proof of the formula for the derivative of a function of a function is to be found therein, and he mentions the careful treatment of the question in Pierpont's *Theory of Functions of a Real Variable* (1905).

It is perhaps worth while to notice that Dini in his *Lezioni di Analisi Infinitesimale* (autographed edition, 1877) and Genocchi-Peano in their *Calcolo Differenziale* (1884) both gave satisfactory proofs. The treatment of Genocchi-Peano is cited and reproduced by Stolz in *Grundzüge der Differential- und Integralrechnung,* Bd. I (1893).

A proof on the same lines as that of Pierpont was given by Tannery in his *Introduction à la Théorie des Fonctions d'une Variable* (1886). See also Cesàro's *Lehrbuch der Algebraischen Analysis und der Infinitesimalrechnung,* Deutsch von Kowalewski (1904), and Kowalewski's *Grundzüge der Differential- und Integralrechnung* (1909).

It is remarkable that even the most careful English writers on the calculus have missed the defect in the proof to be found in our standard works on that subject.

THE UNIVERSITY OF SYDNEY, AUSTRALIA