activities were all in the interests of applied mathematics, yet Klein has continued the directorship and editorship of the MATHEMATISCHE ANNALEN since the death of Clebsch; he completed fifty years in this office last November.

The memoirs on Lamé functions are followed by those on the zeros of the hypergeometric series, the representation of the hypergeometric function by means of definite integrals, and the auto-reviews of the autographed lectures on the hypergeometric function and the linear differential equation of the second order, given in 1893–4. These were all published in the MATHEMATISCHE ANNALEN. The remaining essays were all published elsewhere. They include: a short report on recent English investigations on mechanics, a discussion of space collineations which occur in optical instruments, the greeting given at the opening of the mathematical congress at Chicago, the Princeton sesquicentennial lectures, two papers on graphical statics, one on Painlevé's criticism of Coulomb's law of friction, and finally one on the formation of vortices in frictionless liquids. The list is followed by a detailed explanation of the causes which led to the respective studies. The third and final volume is now in press. It contains the memoirs in the theory of functions.

VIRGIL SNYDER

LÉVY ON FUNCTIONALS

Leçons d'Analyse Fonctionnelle. By Paul Lévy, avec une préface de J. Hadamard. Paris, Gauthier-Villars, 1922. vi + 442 pp.

The increasing importance which is being given to the theory of functionals, or functions of lines, is illustrated by the fact that three of the Borel monographs in the last ten years have been concerned with this branch of mathematics, and the great breadth of the subject is illustrated by the fact that there is so little overlapping between the most recent of these, which is the subject of this review, and the earlier ones by Volterra,* and the more recent Cambridge Colloquium Lectures by Evans. In his introductory chapter, Lévy makes an interesting distinction between "algèbre fonctionnelle" and "analyse fonctionnelle." The first includes problems in which the unknowns are ordinary functions, but where the methods of the theory of functionals are used in determining them. The second includes problems where the unknowns themselves are functions of lines, or where the problems themselves could not be considered independently of the notion of a functional. Most of the work of Volterra and Evans mentioned above would belong to the "algèbre." The present monograph is primarily concerned with the "analyse."

The idea of a continuous functional is of fundamental importance. A functional $U(x(t))$ is said to be continuous if $U(y_n(t))$ approaches $U(x(t))$

* *Leçons sur les Equations Intégrales et les Équations Intégro-differentielles,* reviewed by Westlund in this BULLETIN, vol. 20 (1914), pp. 259–62, and *Leçons sur les Fonctions des Lignes,* reviewed by Bliss, ibid., vol. 21 (1915), pp. 345–55.

as a limit when the "distance" between $y_n(t)$ and $x(t)$ approaches zero. Thus there are as many definitions of a continuous functional as there are definitions of the "distance" between two functions. He mentions several different definitions which have been used, and develops the theory based on two of them to some extent. The definition which he prefers is that the distance r between $x(t)$ and $y(t)$ is given by the formula

$$r^2 = \int_0^1 [y(t) - x(t)]^2 dt.$$

In other words $U(x(t))$ is continuous if $U(y_n(t))$ approaches $U(x(t))$ when $y_n(t)$ approaches $x(t)$ in the mean. Under this definition continuous functionals of the first degree, that is, linear functionals, can always be expressed as Lebesgue integrals, homogeneous functionals of the second degree as double integrals, etc. According to the other definition the distance is the least upper bound of $|y(t) - x(t)|$, and consequently $U(y_n(t))$ does not need to approach $U(x(t))$ unless the sequence $y_1(t)$, $y_2(t)$, $\cdots$ approaches $x(t)$ uniformly, in order to be continuous. This makes it necessary to introduce Stieltjes integrals instead of those of Lebesgue. The author discusses the consequences of this definition under the heading, "le point de vue logique," explaining clearly Hadamard's theorem that a linear functional is the limit of a sequence of definite integrals, F. Riesz' proof that it is also expressible as a Stieltjes integral, and some of Fréchet's work on bilinear functionals. He does not consider this point of view so useful in the applications as "le point de vue pratique," and in the last part of Part I and in the other two parts of the book, he uses the definition based on approach in the mean. The appropriateness of these designations of the two points of view may be questioned, particularly by those who believe that the Stieltjes integral, which is now being used to good advantage by a few applied mathematicians, is destined to be of much greater use in applied science. At the present time, however, the "point de vue logique" appeals particularly to those who are interested in generalizing a mathematical theory for its own sake.

In general Part I is written in a very interesting as well as elementary manner, and is a very valuable introduction to this branch of mathematics.

The second part is mostly the result of the author's own researches, although he refers frequently to the work of Gateaux. The first type of equations considered in Part II is a generalization of equations involving total differentials. The equation

$$du(x_1, x_2, \cdots, x_n) = \Sigma p_i(x_1, x_2, \cdots, x_n, u)dx_i$$

has for its analog

$$\delta U[x(t)] = \int_0^1 f[x(t);\ U, t]\delta x(t)dt,$$

and its conditions of integrability are similar. A system such as

$$\frac{\partial u(x_1, \cdots, x_n, y_1, \cdots, y_n)}{\partial x_i} = f_i\left(x_1, \cdots, x_n, y_1, \cdots, y_n, \frac{\partial u}{\partial y_1}, \cdots, \frac{\partial u}{\partial y_n}, u\right) \qquad (i = 1, 2, \cdots, n)$$

suggests the equation

$$U_x'(x(t), y(t);\ \tau) = f(x(t), y(t), U_y';\ U, \tau),$$

where U_x' is the functional derivative of U considered as a function of $x(t)$, with $y(t)$ entering as a parameter, and U_y' defined similarly. The notions of characteristics, and complete integral, and Cauchy's method are extended to such equations.

If $x(t)$ belongs to a very general class of functionals defined on (0, 1), it can be approached in the mean by a sequence of functions $x_1(t), x_2(t), \cdots$ such that $x_n(t)$ is constant in each of the intervals

$$\left(\frac{r-1}{n} < x \leqq \frac{i}{n}\right).$$

Consequently if $U(x(t))$ is continuous, $U(x_n(t)) \rightarrow U(x(t))$ as $n \rightarrow \infty$. If then x_i is the value of $x_n(t)$ in the ith of these intervals, $U(u_n(t))$ may be considered as a function of n variables and may be called $u_n(x_1, x_2, \cdots, x_n)$. Since

$$\int_0^1 x_n^2(t)dt = \frac{1}{n}\sum_{i=1}^{n} X_i^2,$$

every point inside the sphere in "function space" whose equation is

$$\int_0^1 x^2(t)dt = R^2 \tag{1}$$

can be said to be approached by a sequence of points in the respective n-dimensional spheres

$$\sum_{i=1}^{n} x_i^2 = nR^2, \tag{2}$$

and the $U(x(t))$ will be approached by a sequence of functions $u_n(x_1, \cdots, x_n)$. Also the mean value of $U(x(t))$ in the sphere (1) may be defined as the limit of the mean value of $u_n(x_1, \cdots, x_n)$ in the spheres (2).

Part III begins with a discussion of the sphere in n dimensions, whose radius is $R\sqrt{n}$. If two such spheres have radii $R(1-\alpha)$ and R, where α is an arbitrarily small positive number, the ratio of their volumes will be $(1-\alpha)^n$ which approaches zero as $n \rightarrow \infty$. Thus it may be stated that all the volume of a sphere in function space is located arbitrarily near its surface, excepting a part of relative measure zero. Similarly it is proved that the same volume is concentrated arbitrarily near the equator. It can be proved by means of Stirling's formula that the ratio of the volume of an n-dimensional sphere to the expression $(2\pi e)^{n/2}R^n/\sqrt{n\pi}$ approaches unity as $n \rightarrow \infty$. It follows that the volume approaches zero as $n \rightarrow \infty$ if $R \leqq 1/\sqrt{2\pi e}$ and otherwise becomes infinite. Thus the volume of a sphere in function space may either vanish or be infinite. This does not interfere with the determination of the mean of a functional defined over the sphere, however, as the mean of the corresponding function in n-space may approach a definite limit. The mean is used in generalizing Green's formula, and the equation of Laplace in function space is discussed, and various applications are made.

The preface by Hadamard emphasizes the importance of the subject, and the value of the author's contributions to it.

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