A SET OF AXIOMS FOR LINE GEOMETRY *

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1. *Introduction.* In 1901 Pieri proposed a set of axioms for line geometry in terms of line and intersection.† That Pieri's set of eleven postulates was not independent was shown by Hedrick and Ingold in 1914; they proposed a simpler and more elegant set of but five independent axioms, using the same undefined concepts.‡ Both of these sets are for geometries equiva ent to the general three-space geometry established by axioms A_1, A_2, A_3, E_0, E_1, E_2, E_3 and E_3' of Veblen and Young.§

In this paper is given a set of six independent axioms in terms of line as an undefined element and an undefined class of one-to-one correspondences among the lines called collineations. There is introduced but one defined term before the complete statement of the axioms. To make a proper projective space it has usually been necessary not only to add a postulate of projectivity but also a sequence of definitions for such concepts as perspectivity, projectivity, etc., to give that postulate content. If to our set a seventh postulate is added, we have a proper projective three-space without the intervention of any additional defined concepts.||

2. *Postulates.* Our basis is a class of undefined elements, called *lines*; an undefined class of one-to-one correspondences, or transformations, among the lines, called *collineations*; and

* Presented to the Society, Nov. 27, 1920.

† *Sui principi che regno la geometria delle rette*, TORINO ATTI, vol. 36 (1901), pp. 335–351.

‡ *A set of axioms for line geometry*, TRANSACTIONS OF THIS SOCIETY, vol. 15 (1914), pp. 205–214.

§ *A set of assumptions for projective geometry*, AMERICAN JOURNAL, vol. 30 (1908); *Projective Geometry*, vol. 1, Boston, 1910.

|| Another set of postulates equivalent to the set of all seven of the axioms is the first seven given by the author in his paper *A set of postulates for general projective geometry*, TRANSACTIONS OF THIS SOCIETY, vol. 16 (1915), pp. 51–61.

a class of defined classes of lines, called *fields*. By a one-to-one correspondence among the lines we mean that to every line α there corresponds one and only one line α', and no α' is the correspondent of two distinct lines α_1 and α_2. This concept of one-to-one correspondence may be said to belong to the domain of pure logic; we shall therefore not consider it in our basis as an undefined mathematical notion.* We shall define a field as a class of at least three lines such that: (1) every collineation that makes any two distinct lines of the class correspond to two distinct lines of the class leaves the class invariant; (2) a collineation exists that will make any two distinct lines of the class correspond to any two distinct lines of the class; (3) the class has no proper super-class possessing property (2).

Our axioms, whose technical independence is shown by examples given at the end of this paper, are as follows:

I. *There exists a line.*

II. *If α is a line, there exist five distinct fields containing α.*

III. *If a collineation τ_1 exists such that τ_1 makes correspond to the set of lines A the set of lines B, and if a collineation τ_2 exists such that τ_2 makes correspond to the set of lines A the set of lines C, then a collineation τ_3 exists such that τ_3 makes correspond to the set of lines B the set of lines C.*

IV. *If no collineation exists that makes correspond to the field A the field B, and if no collineation exists that makes correspond to the field A the field C, then a collineation τ exists that makes correspond to the field B the field C.*

V. *If A and B are fields such that a collineation τ exists that makes correspond to the field A the field B, then the fields A and B have a line in common.*

VI. *If A, B and C are fields such that A and B have a line α_1 in common, and B and C have a line α_2 in common, and C and*

* It is true, of course, that no sharp line of demarcation exists between concepts of pure logic and of mathematics; but an author may choose whether or not he will include a given concept in his own discussion. See Federico Enriques, *Problems of Science*, authorized translation by J. Royce. Open Court Publishing Co., 1914, p. 122.

A have a line α_3 in common, then there exists a field D such that D contains the lines α_1, α_2 and α_3.

VII. *If A, B, C and D are four distinct fields having a line α in common, and if collineations τ_1, τ_2 and τ_3 exist such that τ_1 makes correspond to the field A the field B, τ_2 makes correspond to the field B the field C, and τ_3 makes correspond to the field C the field D, then every collineation τ that leaves A, B and C invariant also leaves D invariant.*

3. *Theorems.* THEOREM 1. *If A is a field, then a collineation exists that leaves A invariant.*

Let us assume that no collineation exists that leaves A invariant. If now we let fields B and C of Postulate IV be identical with A, we are led at once to an absurdity.

THEOREM 2. *If a collineation τ exists that makes correspond to the field A the set of lines B (or transforms A into B, in notation, $\tau(A) = B$), then a collineation τ^{-1} exists such that τ^{-1} makes correspond to the set of lines B the field A: $\tau^{-1}(B) = A$.*

By hypothesis, a collineation τ exists such that $\tau(A) = B$, and, by Theorem 1, a collineation τ_1 exists such that $\tau_1(A) = A$. Hence it follows from Postulate III that a collineation τ^{-1} exists such that $\tau^{-1}(B) = A$.

THEOREM 3. *If a collineation τ_1 exists such that τ_1 transforms the field A into a set of lines B, and a collineation τ_2 exists such that τ_2 transforms the set of lines B into a set of lines C, then a τ_3 exists such that τ_3 transforms A into C.*

By Theorem 2, a collineation τ_1^{-1} exists such that $\tau_1^{-1}(B) = A$. The existence of τ_1^{-1} and τ_2, by virtue of Postulate III, implies the existence of a collineation τ_3 such that $\tau_3(A) = C$.

COROLLARY. *The resultant of a sequence of collineations on a field is a collineation.*

THEOREM 4. *Every collineation transforms a field into a field.*

Let τ be any collineation that transforms the field A into a set of lines B. Let β_1, β_2, β_1' and β_2' be any four lines of B such that β_1 and β_2 are distinct and β_1' and β_2' are distinct; and let α_1, α_2, α_1' and α_2' be the four lines of A such that $\tau(\alpha_1, \alpha_2, \alpha_1', \alpha_2') = \beta_1, \beta_2, \beta_1', \beta_2'$. Let τ_1 be any collineation such that $\tau_1(\beta_1, \beta_2) = \beta_1', \beta_2'$. From Theorem 2 and the corollary of Theorem 3, we know that a collineation $\tau_2 = \tau^{-1}\tau_1\tau$ exists such that $\tau_2(\alpha_1, \alpha_2) = \bar{\alpha}_1', \bar{\alpha}_2'$, where $\bar{\alpha}_1'$ and $\bar{\alpha}_2'$ are lines of A. But since the collineation τ_2 makes two lines of the field A correspond to two lines of A (which are distinct, since they are the correspondents of pairs of distinct lines), τ_2 leaves the field A invariant. Therefore, since $\tau^{-1}\tau_1\tau(A) = A$, $\tau(A) = B$, and $\tau^{-1}(B) = A$, we have $\tau^{-1}\tau_1(B) = A$, and hence $\tau_1(B) = B$.

Since A is a field, a collineation τ_3 exists such that $\tau_3(\bar{\alpha}_1, \bar{\alpha}_2) = \alpha_1', \alpha_2'$, where $\tau^{-1}(\beta_1, \beta_2) = \bar{\alpha}_1, \bar{\alpha}_2$. Therefore a collineation $\tau_4 = \tau\tau_3\tau^{-1}$ exists such that $\tau_4(\beta_1, \beta_2) = \beta_1', \beta_2'$.

We have shown that properties (1) and (2) of a field are possessed by the set of lines B. Let us now assume that B has a proper super-class B^* also possessing property (2). If $\tau'(B) = A$ and if $\tau'(B^*) = A^*$, then A^* is a proper super-class of A. Let α_1^*, α_2^* and $\alpha_1^{*\prime}$, $\alpha_2^{*\prime}$ be any two pairs of distinct lines of A^*. Furthermore let β_1^*, β_2^*, $\beta_1^{*\prime}$ and $\beta_2^{*\prime}$ be those four lines of B^* such that $\tau(\alpha_1^*, \alpha_2^*) = \beta_1^*, \beta_2^*$ and $\tau^{-1}(\beta_1^{*\prime}, \beta_2^{*\prime}) = \alpha_1^{*\prime}, \alpha_2^{*\prime}$. By our assumption, a τ_5 exists such that $\tau_5(\beta_1^*, \beta_2^*) = \beta_1^{*\prime}, \beta_2^{*\prime}$; therefore a $\tau_6 = \tau^{-1}\tau_5\tau$ exists. But $\tau_6(\alpha_1^*, \alpha_2^*) = \alpha_1^{*\prime}, \alpha_2^{*\prime}$; hence A^* possesses property (2) of a field, which is impossible if A is a field. Therefore our assumption leads to a contradiction; hence B is a field.

THEOREM 5. *If A and B are distinct fields such that a collineation exists transforming A into B, then A and B have at most one line in common.*

Let us assume that A and B have the two distinct lines γ_1 and γ_2 in common. If the collineation that makes A go over into B is τ, then let $\tau(\gamma_1, \gamma_2) = \beta_1, \beta_2$. Since γ_1, γ_2, β_1 and β_2 are pairs of distinct lines of B, a collineation τ_1 exists such that $\tau_1(\beta_1, \beta_2) = \gamma_1, \gamma_2$ and $\tau_1(B) = B$. But $\tau_1\tau(A) = A$, since γ_1 and γ_2 of A are invariant; and $\tau_1\tau(A) = B$, since $\tau(A) = B$; therefore our assumption must be wrong.

THEOREM 6. *If A is a field containing the lines α_1 and α_2, then there exists a field $\bar{A}$ also containing α_1 and α_2 and such that no collineation exists that will transform A into $\bar{A}$.*

We shall assume α_1 and α_2 distinct, since the theorem is true in the case where α_1 and α_2 are identical if it is true when α_1

and α_2 are distinct. By means of Postulate II we know that there exists a field A_1 containing the line α_1 and distinct from A. Let τ be a collineation such that $\tau(\alpha_1, \alpha_2) = \alpha_2, \alpha_1$. From Theorem 4, we have $\tau(A_1) = A_2$, where A_2 is a field containing α_2. If $A_1 = A_2$, then A_1 is $\bar{A}$, since A and A_1 have the two distinct lines α_1 and α_2 in common; hence they cannot be transformable into one another without contradicting Theorem 5. Let us now assume A_1 and A_2 distinct. Since A_1 is transformable into A_2, fields A_1 and A_2 have a line α in common. But since A and A_1 have the line α_1 in common, A_1 and A_2 the line α in common, and A_2 and A the line α_2 in common, it follows from Postulate VI that a field A' exists containing α, α_1 and α_2. If A is not transformable into A', then A' is $\bar{A}$. If A is transformable into A', then A is identical with A' by Theorem 5. In the latter case, however, A_1 is not transformable into A, since if that were possible, A and A_1 would be identical, since they have the lines α_1 and α (distinct, since otherwise $A_1 = A_2$) in common. Since α, α_1 and α_2 are distinct lines of the field A, a collineation τ_1 exists such that $\tau_1(\alpha_1, \alpha) = \alpha_1, \alpha_2$. The collineation τ_1 transforms the field A_1 into a field A_1' containing the lines α_1 and α_2. But A cannot be transformed into A_1', because, if that were possible, then, by Postulate III, A_1 would be transformable into A. Hence it follows that A_1' is $\bar{A}$ in this case.

THEOREM 7. *Every field belongs to one or to the other of two classes of fields such that collineations exist that will transform any field of one class into any field of the same class, and such that no collineations exist that will transform any field of one class into a field of the other class.*

From Theorem 6, we know that two fields A and $\bar{A}$ exist such that no collineation will transform A into $\bar{A}$. Let $[A_i]$ be the class of all fields transformable into A, and let $[\bar{A}_i]$ be the class of all fields transformable into $\bar{A}$. Let B be any field. If a collineation τ exists such that $\tau(B) = A$, then B belongs to $[A_i]$. Suppose B does not belong to $[A_i]$, then since no collineation exists transforming B into A, and no collineation exists transforming $\bar{A}$ into A, it follows from Theorem 2 and

Postulate IV that a collineation $\bar{\tau}$ exists such that $\bar{\tau}(B) = \bar{A}$; hence B belongs to $[\bar{A}_i]$. But B cannot belong to both $[A_i]$ and $[\bar{A}_i]$, for in that case it would follow from Postulate III that A could be transformed into $\bar{A}$. Any field A_i of the class $[A_i]$ will determine the class, since it is easy to show that if A_i and A_j are both members of the class and hence transformable into A, they are transformable into one another. Furthermore, no A_i of $[A_i]$ can be transformed into an $\bar{A}_j$ of $[\bar{A}_i]$, since that would make A_i belong to both classes.

We shall call the fields of one of the two classes of fields *points*, and the fields of the other class *planes*. We shall use the terminology α is a line *on* the plane p or p is *on* α to mean α is a line of p; P is a point *on* the plane p or p *on* P to mean that P and p have a line in common; P is a point *on* the line α or α *on* P to mean that α is a line of P. If two lines are on same point, they are called *copunctal;* if two lines are on the same plane, they are called *coplanar.*

THEOREM 8. *If a point P is on the line α and α is on the plane p, then P is on p.*

Since P and p have the line α in common, the theorem is obvious.

THEOREM 9. *If two lines are copunctal (coplanar), they are coplanar (copunctal).*

This theorem is but a restatement of Theorem 6 in the light of subsequent definitions.

THEOREM 10. *If P_1 and P_2 are distinct points (planes), then there is at least one line on both P_1 and P_2.*

See Postulate V.

THEOREM 11. *If P_1 and P_2 are distinct points (planes), there is not more than one line on both P_1 and P_2.*

See Theorem 5.

Points (planes) which are on the same line are said to be *collinear;* points (planes) which are not on the same line are said to be *non-collinear.*

THEOREM 12. *If P_1, P_2 and P_3 are three non-collinear points (planes), then there is one and only one plane (point) p such that P_1, P_2 and P_3 are on p.*

Let the line common to P_1 and P_2 be α_3, the line common to P_2 and P_3 be α_1, and the line common to P_3 and P_1 be α_2. By Postulate VI a field exists containing α_1, α_2 and α_3. If this field is a point (plane), then P_1, P_2 and P_3 coincide, contrary to the hypothesis that they are non-collinear and hence distinct. The field is therefore a plane (point). There can be but one such plane (point). Since α_1 and α_2 are distinct (otherwise P_1, P_2 and P_3 are collinear), they determine the plane (point) p. Let α_1' be a line of P_1 in p', α_2' a line of P_2 in p', and α_3' a line of P_3 in p'; hence P_1, P_2 and P_3 are on the plane (point) p'. If α_1' and α_2' coincide, then $\alpha_1' = \alpha_2' = \alpha_3$. Since α_1' and α_2' are coplanar (copunctal), they must be copunctal (coplanar). If α_1' and α_2' are distinct, they determine a point (plane) P_3'. Now P_3' cannot be on α_3 since that would make α_1' and α_2' coincide. By the part of the theorem already proved, a plane (point) p'' exists such that P_1, P_2 and P_3' are on p''. But p'' contains α_1' and α_2'; therefore p'' is p'. Hence the plane (point) p' contains α_3. In like fashion it can be shown that p' contains α_1 and α_2; hence p' is p.

COROLLARY 1. *If P_1 and P_2 are distinct points (planes), and if P_1 and P_2 are both on the plane (point) p, then the line α common to P_1 and P_2 is on P.*

COROLLARY 2. *If P is a point (plane) and α is a line such that P is not on α, then there is one and only one plane (point) determined by P and α.*

THEOREM 13. *If P_1, P_2, P_3 are three points (planes) not on the same line, and α is a line joining P_2 and P_3, the class $[P_i]$ of all points (planes) such that every point (plane) of $[P_i]$ is collinear with P_1 and some point (plane) on α, is such that every point (plane) of $[P_i]$ is on the plane (point) p determined by P and α, and every point (plane) on the plane (point) p is a point (plane) of $[P_i]$.*

Let P_i be a point (plane) collinear with P_1, and $\bar{P}_i$ a point (plane) on α. $\bar{P}_i$ is on p, since $\bar{P}_i$ is on α, by Theorem 8. Let the line determined by P_1 and $\bar{P}_i$ be α_i. Then α_i is on p by Corollary 1, Theorem 12. But P_i is on α_i; therefore P_i is on p.

Let P' be any point (plane) of p. P_1 and P' determine a line α' (unless P' is P_1, in which case P' obviously belongs to $[P_i]$). The lines α' and α are distinct since α is not on P_1, and they are copunctal (coplanar) since both lines are on p. Hence α and α' determine a point (plane) $\bar{P}'$. Hence P' is collinear with P_1, and $\bar{P}'$ a point (plane) on α; consequently P' belongs to $[P_i]$.

THEOREM 14. *If P_1, P_2, P_3 are points (planes) not all on the same line, and P_4 and P_5 ($P_4 \neq P_5$) are points (planes) such that P_2, P_3, P_4 are on a line and P_1, P_3, P_5 are on a line, there is a point (plane) P_6 such that P_1, P_2, P_6 are on a line and also P_4, P_5, P_6 are on a line.*

Since the points (planes) P_1, P_2 and P_3 are non-collinear, they determine a plane (point) p. The three lines α_1 common to P_2 and P_3, α_2 common to P_1 and P_3, and α_3 common to P_1 and P_2, are on p by Corollary 1, Theorem 12. Since P_4 is on α_1, and P_5 is on α_2, P_4 and P_5 are also on p. Hence the line α common to P_4 and P_5 is on p, and consequently α is copunctal (coplanar) with α_1. If α and α_1 are distinct, they determine a point (plane) which is P_6. If α and α_1 coincide, then any point (plane) on α will serve as a P_6.

THEOREM 15. *If α_1 and α_2 are lines, then α_1 and α_2 are on the same number of points (planes).*

Since collineations exist that transform any point (plane) into any point (plane) and any line in a point (plane) into any other in the same point (plane), a collineation τ exists that transforms α_1 into α_2. Every point (plane) on α_1 is transformed by τ into a point (plane) on α_2, and every point (plane) on α_2 comes from a point (plane) on α_1. Hence α_1 and α_2 are on the same number of points (planes).

THEOREM 16. *All points (planes) are not on the same line.*

Let us assume that every point (plane) is on a line α. Let τ be a collineation that transforms α into a distinct line α'. If we transform every point (plane) by τ, we get all the points (planes) again, since distinct fields transform into distinct fields. But all points (planes) now contain α' as well as α. Therefore it follows that there is but one point (plane) P.

Every line is in P, since, by Theorem 6, every line is in a point (plane). No other field can exist, since, if it did, it would be a proper sub-class of P, contrary to the definition of field. But if there is only one field, Postulate II is contradicted. Hence our original assumption is false and the theorem proved.

THEOREM 17. *If α is a line, then the number of planes on α is the same as the number of points on α.*

Since, by Theorem 16, all planes are not on α, there exists a line $\bar{\alpha}$ non-coplanar with α. Every point on $\bar{\alpha}$ determines, with the line α, a different plane, since if two distinct points $\bar{P}_1$ and $\bar{P}_2$ on $\bar{\alpha}$ and α were on the same plane, the lines $\bar{\alpha}$ and α would be coplanar. Hence there are at least as many planes on α as there are points on $\bar{\alpha}$. Likewise every plane on α determines, with the line $\bar{\alpha}$, a different point. Hence there are at least as many points on $\bar{\alpha}$ as there are planes on $\bar{\alpha}$. Since, by Theorem 15, the number of points on α and on $\bar{\alpha}$ are equal, and since the number of planes on α must equal the number of points on $\bar{\alpha}$, it follows that α is on the same number of points as α is on planes.

THEOREM 18. *There are at least three points (planes) on every line.*

By Postulate II, every line is on at least five fields; and, since the number of planes and points on a line is the same, the theorem follows immediately.

THEOREM 19. *All points (planes) are not on the same plane (point).*

Let p be any plane (point). Since there is more than one plane (point), there exists a line α not on p. If two points (planes) of α were on p, then α would be on p. Hence at most one of the points (planes) of α can be on p.

4. *Comparison with Ordinary Geometry.* Let us now compare our geometry with the general three-space geometry of Veblen and Young.* If we identify our lines as lines, collineations as projectivities, and fields as either points or planes in the Veblen-Young geometry, then our axioms are theorems in

* Loc. cit.

their geometry. Incidently, the consistency of our axioms is thus established.

On the other hand, let the undefined *line* of Veblen and Young be our *line*; their undefined *point*, our *point*; and their *on* our *on*. Consider their assumptions of alignment A:

A_1. *If A and B are distinct points, there is at least one line on both A and B.*

A_2. *If A and B are distinct points, there is not more than one line on both A and B.*

A_3. *If A, B, C are points not all on the same line, and D and E ($D \neq E$) are points such that B, C, D are on a line and C, A, E are on a line, there is a point F such that A, B, F are on a line and also D, E, F are on a line.*

We see that, except for notation, A_1 is our Theorem 10, A_2 is Theorem 11, and A_3 is Theorem 14. The *plane* defined by Veblen and Young is equivalent to our *plane*, by virtue of Theorem 13.

The assumptions of extension, substituting for the usual E_3' an equivalent axiom also given by Veblen and Young, are as follows:

E_0. *There are at least three points on every line.*

E_1. *There exists at least one line.*

E_2. *All points are not on the same line.*

E_3. *All points are not on the same plane.*

E_3'. *Any two distinct planes have a line in common.*

E_0 is our Theorem 18; E_1 is Postulate I; E_2 is Theorem 16; E_3 is Theorem 19; and E_3' is Theorem 10. We have therefore proved that our geometry is equivalent to that of Veblen and Young.

The assumption of projectivity is as follows:

P. If a projectivity leaves each of three distinct points of a line invariant, it leaves every point of the line invariant.

It is evident that P is a rephrasing of Postulate VII in terms of point.

5. *Independence Examples.*—1. No lines exist. Hence neither collineations nor fields exist.

—2. Lines are all the lines of a projective three-space and an extra line not in that three-space. Collineations are all the projectivities on the lines of the three-space, but in addition leave the extra line invariant. Fields are therefore the points and planes of the three-space.

—3. Lines are all the lines of a projective finite three-space. Collineations are all the projectivities on the three-space and an extra transformation τ, such that τ transforms every line α into a line α' skew to α. Fields are therefore the points and planes of the three-space.

—4. Lines are all the lines of two projective three-spaces having no line in common. Collineations are all the projectivities on the two three-spaces such that each of the three-spaces is left invariant. Fields are therefore the points and planes of the two three-spaces.

—5. Lines are all the lines of two projective three-spaces having no line in common. Collineations are all the projectivities on the two three-spaces such that each of the three-spaces is left invariant, or such that the two three-spaces are interchanged. Fields are therefore the points and planes of the two three-spaces.

—6. Lines are all the lines of a euclidean plane. Collineations are all the projectivities such that parallel lines are transformed into parallel lines. Fields are therefore pencils of lines.

—7. Lines are all the lines of an improperly projective three-space. Collineations are all the projectivities in such a three-space. Fields are therefore the points and planes of that three-space.

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