THE EVANSTON MEETING OF THE SOCIETY

The fiftieth regular meeting of the Chicago Section of the American Mathematical Society, being the eighteenth regular western meeting of the Society, was held at Northwestern University, Evanston, Illinois, on Friday, December 29, 1922. The attendance at this meeting was about forty-five, among whom were the following thirty-seven members of the Society:

Ballantine, W. S. Beckwith, A. D. Campbell, Chittenden, Coble, Curtiss, H. T. Davis, Dickson, Dowling, Dresden, Edington, Gokhale, Gouwens, Lane, Logsdon, McCain, McClenon, MacMillan, March, Marshall, T. E. Mason, C. N. Mills, E. J. Moulton, Plapp, Schottenfels, Skinner, J. H. Taylor, Theobald, E. L. Thompson, B. M. Turner, W. J. Wagner, R. L. Wilder, K. P. Williams, R. E. Wilson, F. E. Wood, Yanney, Zehring.

At the business meeting on Friday afternoon a resolution, proposed by Professor Skinner, expressing the Society's appreciation of the reception accorded them at Northwestern University was unanimously adopted. On Friday evening a dinner was held at the North Shore Hotel at which twenty-five persons were present. Toasts were responded to by Professors Holgate, Dickson, Curtiss, and Dresden.

At the sessions on Friday Professor Coble, Chairman of the Section, presided, relieved by Professor Curtiss in the forenoon and by Professor Dickson in the afternoon. Titles and abstracts of the papers read at this meeting are given below. Professor Moore's paper was presented by Mr. Wilder. The papers of Professor Jackson and Dr. Camp, and Professor Chittenden's first paper were read by title.

1. Professor E. P. Lane: *Ruled surfaces with generators in one-to-one correspondence.*

The author bases a projective theory of ruled surfaces, whose generators are in one-to-one correspondence and skew to each other, on a system of four ordinary linear first order differential equations in four dependent variables. This system is reduced to a canonical form by referring the surfaces to their intersector curves. The intersector curves, which are analogous to asymptotics, and certain other curves analogous to flecnode curves, are studied.

The general theory is applied in the investigation of Green-reciprocal ruled surfaces. Several of their properties are developed. In particular should be mentioned a characterization of the directrix congruences in terms of simple geometrical concepts.

2. Mr. R. L. Wilder: *Some theorems on continuous curves, with special reference to continuous curves that contain no simple closed curve.*

Among the theorems established are: (I) A sufficient* condition that a continuum M should be a continuous curve is that every two points that lie in a connected subset of an open subset of M lie in a sub-continuum of that open subset; (II) If N is a closed proper subset of a continuous curve M, then $M - N$ is a countable set of domains with respect to M, no two of which have a point in common, and if M contains no simple closed curve, then for $\epsilon > 0$, at most a finite number of these domains is of diameter $> \epsilon$; (III) A continuous curve M that contains no simple closed curve consists of (1) a sequence of arcs $C_1, C_2, C_3, \cdots$ every two of which have in common at most a point which is not interior to both, and such that (i) n being any positive integer, $C_1 + C_2 + \cdots + C_n$ is a continuous curve M_n and a proper subset of M, (ii) for $\epsilon > 0$ there exists a number ρ such that for $n > \rho$, $\delta(C_n) < \epsilon$ (δ meaning "diameter of") and $\delta(d) < \epsilon$, where d is any one of the maximal domains with respect to M lying in $M - M_n$; and (2) a totally disconnected set of non-cut points, P_ω, each of which is a limit point of the sequence $M_1, M_2, M_3, \cdots$, and containing all the non-cut points of M.

3. Mr. R. L. Wilder: *An analysis of the point set which constitutes the boundary of a complementary domain of a continuous curve.*

It is shown that the boundary β of a complementary domain of a continuous curve is the sum of, at most, a countable set of arcs and simple closed curves together with a totally disconnected set of limit points. It is also shown that: (I) Every closed and connected subset of β is itself a continuous curve, (II) β cannot contain, for any positive number ϵ, an infinite set of arcs of diameter $> \epsilon$, such that every two of

* R. L. Moore has proved this condition necessary. See this BULLETIN, vol. 28 (1922), p. 91.

these arcs have in common at most a point which is not an interior point of both, (III) If N is any connected subset of β, then every two points of N are the extremities of an arc lying wholly in N, (IV) A necessary and sufficient condition that the boundary of a simply connected domain be a continuous curve is that every connected subset of it should be connected in the strong sense.

4. Professor R. L. Moore: *An uncountable non-dense closed point set each of whose complementary intervals abuts on another one at each of its ends.*

This paper appeared in full in the February number of this BULLETIN.

5. Dr. Mayme I. Logsdon: *Closed sets of rational points on a plane cubic curve of genus one.*

If the elliptic argument of a rational point, A, on a plane cubic curve of genus one is commensurable with a period of the functions of the parameter which enter in the coordinate representation of the point, the process of using A to find other rational points by the method of tangentials and secants will terminate after a finite number of such points have been found. The author has studied certain configurations which may arise, in particular when the anharmonic ratio of the cubic is rational.

6. Mr. H. T. Davis: *Report on a boundary value problem of fourth order.*

The problem studied is that of developing an existence theorem for the characteristic numbers λ_i of the system $L(u) + \lambda k(x)u = 0$, $U_i(u) = 0$ $(i = 1, \cdots, 4)$, where $L(u)$ is the most general self-adjoint expression of fourth order, the range of x is between two singular points and $U_i(u)$ are self-adjoint linear expressions in u, u', u'', u''', two of them being taken for $x = a$ and two for $x = b$. Real transformations of independent and dependent variables reduce the problem to a normal form, preserving the self-adjointness of the system. A criterion from the theory of integral equations is then applied and an existence theorem for the λ_i developed. The problem is shown to be the analogue of Sturm's boundary value problem for a system of second order. While the solution of the latter problem may be shown to depend upon the oscillation of a certain solution of an equation of second order, the solution in the present case depends upon the oscillation of a certain solution of a fifth order equation due to Halphen.

7. Professor A. B. Coble: *The extension of the Weddle and Kummer surfaces to hyperelliptic 3-ways of genus 3.*

The author has previously shown that the generalized Kummer surface for $p = 3$, an M_3^{24} in S_7, can be defined as the double manifold of a certain quartic locus in S_7 whose equation has coefficients which are rational in the coordinates of an Aronhold set of 7 double tangents of the normal curve of genus 3—a ternary quartic. This definition fails completely for the hyperelliptic case since then the normal curve is a ternary quintic with a triple point. To cover this case the Kummer surface is obtained as a birational transform of a generalized Weddle surface—a 3-way in S_5—which is defined by 8 points in S_5 as the ordinary Weddle is defined by 6 points in S_3, i.e., as the locus of fixed points of a Cremona involution attached to the set of points.

8. Professor A. B. Coble: *Associated sets of points.*

A set of n points in S_k, P_n^k, is *associated* with a set Q_n^{n-k-2} if complementary minors of the matrices of the coordinates of the two sets are proportional. The author considers various methods of mapping the one set upon its associated set or of projecting the one upon the other. Properties of the set in the higher dimension are thus deduced from known properties of the associated set.

9. Professor L. E. Dickson: *The rational linear algebras of maximum and minimum ranks.*

The investigation relates to linear associate algebras with a principal unit, the coordinates of whose numbers range over an arbitrary field F. Such an algebra in n units of rank n is irreducible with respect to F if and only if $f(x)$ is a power of a polynomial irreducible in F, where $f(x) = 0$ is the rank equation. An algebra over F, of rank 2, has units $1, e_1, \cdots, e_m$ such that $e_i^2 = c_i$, $e_ie_j = -e_je_i$ $(i \neq j)$, where c_i belongs to F. If at least two c_i are not zero, then $m = 3$ and the algebra is the rational generalization of quaternions. If $c_i = 0$ $(i < m)$, $c_m \neq 0$, the algebra is equivalent either to

$$e_m^2 = 1, \qquad e_me_i = e_i, (i \leqq k), \qquad e_me_j = -e_j, (k < j < m),$$
$$e_re_s = 0, (r < m, s < m,)$$

where $2k \geqq m - 1$, or to

$e_m e_{2i-1} = e_{2i}, \quad e_m e_{2i} = c_m e_{2i-1}, (i \leqq r), e_j e_k = 0, (j < m, k < m),$ with $m = 2r + 1$, and c_m not a square in F. Finally, if every $c_i = 0$, the square of every number of the algebra is zero (a case of the outstanding problem of nilpotent algebras), and the rather numerous algebras in fewer than eight units are found. These results are applied to the determination of all algebras over any field F in 2, 3 or 4 units.

10. Professor L. E. Dickson: *A new simple theory of hypercomplex integers.*

The definition of a system of hypercomplex integers due to A. Hurwitz and applied to all classic algebras in 2, 3 and 4 units by Du Pasquier postulates rational coordinates, a finite arithmetical basis, closure under addition, subtraction and multiplication, the presence of the n basal units $e_0 = 1, \cdots, e_n$ (or only of e_0), and that the system is a maximal. Unfortunately, no maximal system exists for the majority of algebras. If we employ any non-maximal system, it usually happens that factorization into primes is not unique and cannot be made unique by the introduction of ideals of any kind. These essential difficulties all disappear if we replace the postulate of a finite basis by the assumption that, for every number of the system of integers, the coefficients of the rank equation are all rational integers.

11. Professor Arnold Dresden: *Symmetric forms in n variables.*

This paper considers the problem of expressing a symmetric form in n variables of weight S, of k parts and of given weight of each part, as a polynomial in the elementary symmetric functions of these variables. By means of a dominance relation between partitions of S, it is possible to specify which products of elementary symmetric functions will be involved. Various theorems are given for the determination of the coefficients with which these products will appear in the final result.

12. Professor Dunham Jackson: *A general class of problems in approximation.*

The results of recent papers by the author on the method of least mth powers are extended by the admission of a more general function of the error in place of the mth power. If y denotes the error, $|y|^m$ is replaced by a function $E(y)$ which is defined and continuous for all real values of y, becomes positively infinite as y becomes infinite in either direction, and, in most of the theorems, possesses additional properties, varying from one theorem to another. Among the topics treated are the existence and uniqueness of the approximating function, the continuity of its dependence on the function for which an approximate representation is sought (a question which was not discussed explicitly in the earlier papers mentioned), and, in the case of approximation by finite trigonometric sums, the convergence of the approximating sum as its order is indefinitely increased. Some of the results are obtained by fairly direct application of the methods previously used; others call for new considerations of some delicacy.

13. Dr. W. E. Edington: *Abstract group definitions and applications.*

Three distinct infinite systems of non-abelian groups defined by the conditions $t_1^4 = t_2^4 = (t_1t_2)^2 = (t_1t_2^3)^\alpha = 1$, $t_1^3 = t_2^6 = (t_1t_2)^2 = (t_1^2t_2^2)^\alpha = 1$, and $t_1^3 = t_2^3 = (t_1t_2)^3 = (t_1t_2^2)^\alpha = 1$, are proved to exist for all values of α and the generators as substitutions are given. The groups are of order $4\alpha^2$, $6\alpha^2$, and $3\alpha^2$, respectively, and are all solvable. The first two systems are the groups of the elliptic norm curve C_α, respectively, when the invariants g_3 and g_2 are zero. The principal subgroups and the orders of the operators are determined for the groups in each system.

14. Dr. W. E. Edington: *On an infinite system of non-abelian groups of order nm^{n-1}.*

If two non-commutative operators s_1 and s_2 fulfil the conditions

$$s_1^n = s_2^n = (s_1^\alpha s_2^\alpha)^m = 1, \qquad s_1^\alpha s_2^\alpha s_1^\beta s_2^\beta = s_1^\beta s_2^\beta s_1^\alpha s_2^\alpha, \quad (\alpha, \beta = 0, 1, 2, \cdots, (n-1)),$$

they generate a group of order nm^{n-1}, and such a group exists for all values of m and n. The generators as substitutions are given in the paper. The groups of this system are all solvable. Several known infinite systems are special cases of this more general system.

15. Dr. W. E. Edington: *On an infinite system of non-abelian groups of order nm^n.*

Two non-commutative operators s_1 and s_2 having the property that the transforms of their product s_1s_2 by s_1 and its powers are all distinct and commutative, and also fulfilling the conditions $s_1^n = s_2^{mn} = (s_1s_2)^m = 1$, generate a group of order nm^n, and groups exist for all values of m and n. The generators represented as substitutions also possess interesting properties.

16. Professor E. W. Chittenden: *Note on a property of abstract sets which admit a definition of distance.*

Fréchet gives the following definition of a complete set (D): a set (D) with preassigned elements of accumulation is *complete* if among all possible definitions of distance in (D) for which the elements of accumulation remain unchanged there is one with respect to which (D) admits a generalization of the theorem of Cauchy regarding the convergence of sequences; and remarks that it would be interesting to know whether there exist sets (D) which are not complete. It is shown that if a set (D) contains a set H which is in its derived set H', then (D) contains a perfect set. Therefore the rational numbers with distance defined as usual form a set (D) which is not complete.

17. Professor E. W. Chittenden: *The Schmidt linear differential forms of a limited bilinear form in infinitely many variables.*

The Hellinger theory of the linear differential forms of a quadratic form in infinitely many variables can be used to obtain a representation of the general limited bilinear form which is analogous to Schmidt's representation of the unsymmetric kernel $K(x, y)$ in terms of the fundamental functions of $\overline{K}(x, y) = \int_a^b K(x, t)K(y, t)dt$, $\underline{K}(x, y) = \int_a^b K(t, x)K(t, y)dt$.

18. Professor E. W. Chittenden: *On a form of the property of Borel-Lebesgue which is independent of the closure of derived classes.*

The closure of derived classes is extensionably attainable in terms of the concept, "limiting element of transfinite order." The property of Borel-Lebesgue and the property, "perfectly compact," of R. L. Moore, may be so generalized that the new properties are equivalent in any system (V) of Fréchet which has the property 2°, that any element of accumulation of the sum of two classes is an element of accumulation of at least one of the two classes. Furthermore, if derived classes are closed (in the usual sense), the generalized properties are equivalent, respectively, to the property of Borel-Lebesgue and the property, "perfectly compact."

19. Professor K. P. Williams: *Concerning an expansion in the restricted problem of three bodies.*

In one of the Lagrange particular solutions of the problem of three bodies, the function $y(\mu)$ defined by

$$y^5 + (3 - \mu)y^4 + (3 - 2\mu)y^3 - \mu y^2 - 2\mu y - \mu = 0$$

comes into consideration (Moulton, *Celestial Mechanics*, 1st edition, p. 197). The purpose of the present paper is to determine the radius of convergence of the series that gives y.

20. Dr. C. C. Camp: *Expansions in terms of solutions of partial differential equations.* Second paper.

The author considers the equation $\sum_{i=1}^{p} L_i(u) + \lambda u = 0$ with the boundary conditions $T_{ij}(u) = 0$, $i = 1, 2, \cdots, p$; $j = 1, 2, \cdots, n_i$, where $L_k(u)$ is a partial differential expression of order n_k in x_k similar to the ordinary differential expression in x considered by Birkhoff in his thesis, and $T_{kj}(u)$ is a partial differential expression of order $n_k - 1$ such that $T_{kj}(u) = 0$ reduces to one of Birkhoff's *regular* boundary conditions in a_k and b_k by the transformation $u = p_{i=1}^{\rho} u_i(x_i)$. This gives rise to a set of p Birkhoff systems, each involving a different parameter. By employing a convergence proof similar to that used in his first paper the author shows that a function f in p variables, similarly restricted in the region $a_i \leqq x_i \leqq b_i$, may be expanded analogously in a *multiple Birkhoff series* in terms of principal solutions of the set of systems and of the adjoint set. Such a series has the same convergence properties as the multiple Fourier expansion except at points on the boundaries. The development has applications in important physical problems such as those of potential, heat flow, and wave motion.

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