SHORTER NOTICES

Vorlesungen über Zahlen- und Funktionenlehre. Erster Band, dritte Abteilung. By Alfred Pringsheim. Leipzig and Berlin, B. G. Teubner, 1921. ix + 515–976 pp.

The first part of this volume was reviewed in a previous number of this BULLETIN (vol. 25 (1919), p. 470). The second part, which appeared shortly after the first, was devoted to a detailed exposition of the theory of infinite series with real terms. The third part treats of complex numbers, infinite series with complex terms, infinite products, and continued fractions. It also contains an appendix to the whole first volume and an index. The appendix (pp. 917–969) consists of numerous references to the literature and indications of the historical development of portions of the theory, as well as further discussion of many of the topics treated in the body of the text.

The exposition exhibits Professor Pringsheim's characteristic lucidity, and the theory throughout is developed from first principles in elementary fashion. By the phrase "elementary fashion" we do not wish to connote anything in the nature of looseness of treatment or lack of rigor, for the author has been at particular pains to avoid defects of that sort. Thus the book is well adapted to meet the needs of any reader, regardless of the extent of his previous mathematical knowledge, who wishes to have a complete and logically accurate account of the arithmetic foundations of modern analysis.

In spite of the elementary treatment, the book is in no way limited to the elements of the subject and in connection with various particular topics comes close to the confines of our present-day mathematical knowledge. For example in the second chapter of Section III,* as a preparation for the treatment of the multiplication of series, the methods of Hölder and Cesàro for summing divergent series are introduced and the equivalence of these two methods, a theorem of comparatively recent date, is established. In this connection it is of interest to note that Professor Pringsheim objects to the use of the phraseology "summation of a divergent series" and introduces in place of it the expression "reduction (Reduktion) of a divergent series." He thus speaks of a certain series as being reducible (reduzibel) with a particular associated limit (zugeordneter Grenzwert). He bases his objection to the current usage on the statement that at the present time there exists no precise definition of the term "sum of a divergent series," and in his opinion an epithet so expressive and sonorous (ein so prägnant klingendes Beiwort) as summable should have a precise and definite meaning, such as the words convergent and divergent possess.

It will be recalled by those particularly interested in the study of divergent series that the objections raised by Professor Pringsheim to the present use of the word summable are quite similar to those previously expressed by Professor W. B. Ford in his book *Studies on Divergent*

* The whole first volume has been divided into four principal sections, of which Part three contains Sections III and IV.

Series and Summability and his address entitled *A conspectus of the modern theory of divergent series,* published in volume 25 of this BULLETIN. It will also be recalled that the remedy proposed by the latter was more drastic, namely to give the word summable a definite meaning, based on the notion of analytic extension, and thus exclude from consideration any divergent series that did not satisfy the proposed definition of summability. A combination of the suggestions of Professors Ford and Pringsheim, namely the use of Professor Ford's definition to set off a definite class of divergent series to be known as summable series, and the use of Professor Pringsheim's term reducible in connection with other divergent series might be more generally acceptable than either suggestion alone. It is the opinion of the reviewer, however, that the present use of the word summable, even if it does involve a certain vagueness in the meaning of such an excellent word, is rather too firmly entrenched to be changed overnight. But the objections of Professors Ford and Pringsheim do have a certain validity, and regardless of the opinion of particular individuals the final terminology will probably be an illustration of the survival of the fittest.

In connection with the introduction of complex numbers in Chapter I of Section III, the author departs from the generally adopted method of first building up a system of pairs of real numbers with arbitrarily assigned rules of combination. He begins by introducing the pure imaginary as a new type of number adapted to provide a solution for equations of the type $x^2 = - a^2$ (a real). Then after developing the properties of this new system on the basis of the principle that in their rules of combination they should obey the fundamental laws of algebra, he leads naturally to the introduction of complex numbers by showing that the result of adding a real number and an imaginary number will not be a number of either of these types.

The treatment of continued fractions, to which all of Section IV is devoted, is particularly complete. As regards questions of convergence it contains practically all of our present-day knowledge of the subject. In his preface the author expresses the hope that this part of the book will serve to gain for this highly important, and in his opinion not sufficiently appreciated, topic new devotees from the ranks of mathematical novitiates.

The book proper contains relatively few references to the literature, these having been in the main relegated to the appendix. This arrangement is in line with the purpose of making the treatment particularly suitable for beginners, since it avoids undue interruption in the orderly presentation of the theory and the possibly forbidding aspect of numerous footnotes and citations in the body of the text. With regard to the appendix the author specifically disclaims any pretensions to completeness and excuses himself in advance in case he may on this account have passed over priority claims of particular individuals. In this connection he expresses the opinion that the custom of attaching the names of the discoverers to mathematical theorems has been carried too far in recent times and has led to what he considers a depreciation of the value of mathematical immortality. For this reason he has in general avoided the use of the name of the originator in connection with a theorem or a method except in instances where the usage may be regarded as classical. One is tempted to inquire, however, how these classical usages could have originated if some one had not initially honored an author by naming one of his discoveries after him? It would seem that the whole question reduces to a matter of taste, as to whether one prefers the personal or the impersonal point of view in the study of mathematical science.

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Theorie des Potentials und der Kugelfunktionen. By A. Wangerin. Band II, Sammlung Schubert LIX. Berlin and Leipzig, Vereinigung Wissenschaftlicher Verleger, 1921. viii + 286 pp.

The first volume of the above work, which deals with the fundamental portions of the potential theory, was reviewed in a previous number of the BULLETIN (vol. 16 (1910), p. 492). The present volume is devoted to a study of the properties of spherical harmonics and their applications to various problems of potential theory.

The book is divided into four sections. The first section deals with such properties of spherical harmonics as are essential for later developments; the second deals with potential problems for the sphere; the third with potential problems for the ellipsoid of revolution, for two spheres, and for a few other special cases; the fourth with potential problems for arbitrary closed surfaces.

The book as a whole contains more material than one would expect to find in a work of its size, and the presentation is in general clear and sufficiently refined for the purpose in hand. There are, however, certain errors and inconsistencies in the statements and demonstrations of some of the fundamental theorems that should be eliminated. For example, any one familiar with the literature on spherical harmonics will be surprised to find on page 93 the claim that the author has established the convergence of the development in Laplace's functions of any function $f(\theta, \varphi)$ that is finite, single-valued, and continuous on the whole sphere. Since examples have been constructed by Haar and Gronwall of continuous functions whose development in spherical harmonics is divergent, one naturally examines the "proof" of the so-called theorem with some curiosity. On page 89 he finds the assumption that a sum of p terms, each one of which approaches zero, will also approach zero, even though p becomes infinite at the same time that the individual terms approach zero. On page 93 he finds a statement that is equivalent to the assumption that a function continuous throughout a certain interval does not have an infinite number of maxima and minima in that interval.

It would surely be better to omit entirely any proof of the development theorem, important as it is for the subsequent theory, than to introduce such obvious errors with regard to the fundamentals of analysis in the course of the demonstration. A less radical remedy is available, however, for by adding a further restrictive condition to the statement of the theorem and a relatively small amount of material to the proof, the whole discussion could be put on an entirely rigorous basis. It is to be hoped that these changes will be made, if a subsequent edition appears.

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