A GENERALIZATION OF NORMAL CONGRUENCES OF CIRCLES *

BY J. L. WALSH

1. *Introduction.* A congruence of circles in three-dimensional space is said to be *normal* if every circle of the congruence is normal to three surfaces. Normal congruences have long been studied,† and one of their principal properties is expressed in the theorem that if a variable circle C is normal to the three fixed surfaces S_1, S_2, S_3 at the points P_1, P_2, P_3 respectively, and if the point P_4 is determined by a real constant cross ratio with P_1, P_2, P_3, then as C varies the point P_4 traces a surface which is also orthogonal to C.

It is the purpose of the present note to consider a type of congruence to which we shall give the name of isogonal congruence and which is a generalization of the notion of normal congruence. A congruence of circles is said to be *isogonal* if every circle of the congruence cuts three surfaces at equal angles in such a way that when the circle is inverted into a straight line L, the tangent planes to the corresponding surfaces at their points of intersection with L are all parallel. That is, every sphere through a circle of the congruence cuts at equal angles the three surfaces at their points of intersection with that circle. It is to be noted that the term isogonal might well be given to a still larger type of congruence of circles, but in the present paper the term will be used only in the restricted sense indicated.

We shall prove (Theorem III) that if a congruence is isogonal there are not merely three surfaces but a one-parameter family of surfaces which have the isogonal property, and all the surfaces of the family can be obtained as in the case of normal congruences.

* Presented to the Society, December 27, 1922.

† By Ribaucour, Darboux, Bianchi, Eisenhart, and Coolidge, among others. Detailed references are given by Coolidge, *A Treatise on the Circle and the Sphere*, Oxford, 1916, Chap. XV.

Isogonal congruences of circles are thus a generalization of normal congruences of circles, of normal congruences of lines, and of certain congruences of lines which naturally arise in connection with the parallel mapping of surfaces*. Isogonal congruences are particularly interesting because in general three arbitrary surfaces determine one or several such congruences isogonal to them. If one fixed circle C is isogonal to three surfaces, there is in general one and only one congruence of circles isogonal to those surfaces and containing C. For the condition of isogonality for a circle C' and three surfaces C_1, C_2, C_3 is satisfied if two spheres through the circle C' cut at equal angles the surfaces C_1, C_2, C_3 at their points of intersection with C'. This is equivalent to four independent conditions on all the six-parameter family of circles in space.

Let us proceed to investigate the analogue in the plane of the isogonal congruence.

2. *Isogonal Series in the Plane.* The name *isogonal series* shall be given to a one-parameter family of circles in the plane such that each circle of the family cuts at equal angles three curves, and in such a way that when the circle is inverted into a straight line, the tangents to the transformed curves at their points of intersection with the transformed circle are all parallel. In general, three arbitrary curves determine one or more isogonal series, and if a circle C cuts isogonally three curves there is in general one and only one isogonal series which contains C and all of whose circles cut isogonally those three curves. For the condition of isogonality to three curves involves two independent conditions on the three-parameter family of all circles of the plane. We shall proceed to prove the following theorem.

THEOREM I. *Let a variable circle C cut three fixed curves C_1, C_2, C_3 isogonally at the variable points z_1, z_2, z_3 respectively. Then the point z_4 defined by the real constant cross ratio*

$$(1) \qquad \lambda = (z_1, z_2, z_3, z_4) \equiv \frac{(z_1 - z_2)(z_3 - z_4)}{(z_2 - z_3)(z_4 - z_1)}$$

* Isogonal congruences are also a generalization of a type of congruences of circles considered by F. W. Beal, ANNALS OF MATHEMATICS, (2), vol. 17 (1915–16), pp. 180–170.

traces a curve C_4 *such that the circle* C *cuts isogonally* C_1, C_2, C_3, C_4 *at the points* z_1, z_2, z_3, z_4.

We fix our attention on a particular circle C and its points of intersection z_1', z_2', z_3' (supposed distinct) with C_1, C_2, C_3. We shall prove that *if* z_1, z_2, z_3 *move simultaneously from* z_1', z_2', z_3' *along* C_1, C_2, C_3 *in any way whatever, then the point* z_4 *defined by* (1) *traces a curve which* C *cuts isogonally with* C_1, C_2, C_3.

If z_1 moves from z_1' along C_1, but z_2 and z_3 are kept coincident with z_2' and z_3', then z_4 moves from z_4' along a curve which is cut by C isogonally with C_1. This becomes obvious if z_3' is transformed to infinity; equation (1) then represents a transformation (z_1, z_4) of similitude with z_2' as center, while C is a straight line which is unchanged by the transformation. Likewise if z_2 moves from z_2' along C_2, but z_1 and z_3 are kept coincident with z_1' and z_3', then z_4 moves from z_4' along a curve which is cut by C isogonally with C_2. A similar fact obtains if z_3 moves from z_3' along C_3. Thus, independent infinitesimal changes of z_1, z_2, z_3 from z_1', z_2', z_3' along C_1', C_2', C_3' move z_4 from z_4' along a curve of the kind described, so simultaneous infinitesimal changes of z_1, z_2, z_3 from z_1', z_2', z_3' along C_1', C_2', C_3' also move z_4 from z_4' along a curve C_4 such that C cuts the curves C_1, C_2, C_3, C_4 isogonally at z_1', z_2', z_3', z_4'.* This completes the proof of Theorem I.

Theorem I is particularly interesting in the case that C_1,

* The detailed analysis of the differentials involved is extremely simple in the present case. We have by differentiation of (1) and substitution of (1) in the result,

$$(1') \quad dz_4 = \frac{(z_1-z_2)dz_3+(z_3-z_4)(dz_1-dz_2)-\lambda(z_2-z_3)dz_1-\lambda(z_4-z_1)(dz_2-dz_3)}{(z_1-z_2)+\lambda(z_2-z_3)}.$$

Transform the circle C of Theorem I into the axis of reals, identify the real values z_i' of Theorem I with the z_i of $(1')$, and identify the variables z_i of Theorem I with the $z_i + dz_i$ of $(1')$. All the quantities in $(1')$ except the differentials are real; all the differentials of the right-hand member have the same argument (mod π), so dz_4 has also that same argument (mod π).

Thus if $z_1(t)$, $z_2(t)$, $z_3(t)$, $z_4(t)$ are solutions of a Riccati equation whose cross ratio is a real constant, then whenever the $z_i(t)$ vary as functions of t, they trace paths that are cut isogonally by the circle on which the $z_i(t)$ lie.

C_2, C_3 are all circles. In this case we have the following theorem.*

THEOREM II. *Let C_1, C_2, C_3 be three fixed non-coaxial circles. Then the circles C which cut isogonally C_1, C_2, C_3 at points z_1, z_2, z_3 form four distinct series,† each of which is composed of the circles of a coaxial family. If there are considered the circles C of but one isogonal series, the point z_4 defined by the real constant cross ratio*

$$\lambda = (z_1, z_2, z_3, z_4)$$

traces a circle C_4 which is such that C cuts isogonally C_1, C_2, C_3, C_4 at z_1, z_2, z_3, z_4.

If C_1, C_2, C_3 are coaxial but not all tangent at a single point, there is but one series of circles C cutting them isogonally, namely the circles of the coaxial family conjugate to the family to which C_1, C_2, C_3 belong, and all these circles C cut the three given circles orthogonally. But the points z_1, z_2, z_3 may be chosen on C and on the circles C_1, C_2, C_3 in four essentially different ways, always so that C cuts C_1, C_2, C_3 isogonally at z_1, z_2, z_3. Thus we still have four circles C_4 and for any particular choice of C_4, the circle C cuts isogonally C_1, C_2, C_3, C_4 at z_1, z_2, z_3, z_4.

If C_1, C_2, C_3 are all tangent at a single point P, any circle C through P cuts the original circles isogonally at the intersections of C and those circles distinct from P, so no isogonal series is defined. We can still obtain the four circles C_4, however, by requiring respectively (1) that no point z_1, z_2, z_3 shall coincide with P; (2) that z_1 shall always lie at P; C must then be orthogonal to C_1, C_2, C_3; (3), (4) similarly for z_2 and z_3. Always the circle C_4 is traced by the point z_4, and the circle C cuts C_1, C_2, C_3, C_4 isogonally at z_1, z_2, z_3, z_4.

3. *Isogonal Congruences in Space.* Theorem I and its proof

* See Walsh, TRANSACTIONS OF THIS SOCIETY, vol. 22 (1921), pp. 101–116; Lemma IV.

† By a proper convention for the *angle between two circles,* these four systems are described respectively by saying that C cuts C_1, C_2, C_3 all at the same angle or one of those circles at an angle supplementary to the angle cut on the other two. A similar remark obtains below for Theorem IV.

as just given extend directly to space. Let us prove the following theorem.

THEOREM III. *Let a variable circle C cut isogonally three fixed surfaces C_1, C_2, C_3 at the variable points P_1, P_2, P_3. Then the point P_4 defined by the real constant cross ratio*

$$\lambda = (P_1, P_2, P_3, P_4) \tag{2}$$

traces a surface C_4 such that the circle C cuts isogonally C_1, C_2, C_3, C_4 at the points P_1, P_2, P_3, P_4.

We fix our attention on a particular circle C and its points of intersection P_1', P_2', P_3' (supposed distinct) with C_1, C_2, C_3. We shall prove that *as P_1, P_2, P_3 move from P_1', P_2', P_3' on C_1, C_2, C_3 in any way whatever, then P_4 as defined by* (2) *traces a surface C_4' such that C cuts isogonally C_1, C_2, C_3, C_4' at P_1', P_2', P_3', P_4', where P_4' is defined by $\lambda = (P_1', P_2', P_3', P_4')$.*

If P_1 moves from P_1' along C_1, but P_2 and P_3 are kept coincident with P_2' and P_3', then P_4 traces a surface which is cut by C at P_4' isogonally with C_1, C_2, C_3 at P_1', P_2', P_3'. The corresponding fact holds if P_2 or P_3 is allowed to move on C_2 or C_3 while the other two of the original three points are kept fixed. Independent infinitesimal changes of P_1, P_2, P_3 from P_1', P_2', P_3' along C_1, C_2, C_3 therefore move P_4 along a surface of the kind described, so simultaneous infinitesimal changes of these points must move P_4 along a surface C_4' such that C cuts isogonally C_1, C_2, C_3, C_4' at P_1', P_2', P_3', P_4'. Thus even if a congruence is not isogonal but a single circle C of the congruence cuts isogonally the surfaces C_1, C_2, C_3, then C cuts isogonally with C_1, C_2, C_3 the surface traced by the point P_4 defined by (2).

We leave to the reader the proof of the following theorem, which is the space analogue of Theorem II.

THEOREM IV. *Let C_1, C_2, C_3 be three fixed non-coaxial spheres. Then the circles C which cut isogonally C_1, C_2, C_3 at points P_1, P_2, P_3 form four distinct congruences, each of which is composed of the circles through two points, real, coincident, or imaginary. If there are considered the circles C of but one isogonal congruence, the point P_4 defined by the real constant* *cross ratio*

$$\lambda = (P_1, P_2, P_3, P_4)$$

traces a sphere C_4 *which is such that* C *cuts isogonally* C_1, C_2, C_3, C_4 *at* P_1, P_2, P_3, P_4.

If the spheres C_1, C_2, C_3 are coaxial but not all tangent at a single point, there is but one congruence of circles cutting them isogonally; all of these circles C cut C_1, C_2, C_3 orthogonally. However, the points P_1, P_2, P_3 may be chosen on C and on their proper spheres in four essentially different ways, and in each case C cuts isogonally C_1, C_2, C_3 at P_1, P_2, P_3. Thus the point P_4 still traces four spheres C_4; and for any particular C_4, the circle C cuts isogonally C_1, C_2, C_3, C_4 at P_1, P_2, P_3, P_4.

If the spheres C_1, C_2, C_3 are all tangent at a single point P, any circle C through P cuts those spheres isogonally at the intersections of C and these spheres distinct from P, so we have no unique isogonal congruence. There are, however, four spheres C_4 which can be obtained by requiring respectively that P_1, P_2, P_3, or that none of those points should coincide with P. In the former cases the circle C must be orthogonal to C_1, C_2, C_3 to have the proper isogonal property; in the latter case all the circles C to be considered form a complex instead of a congruence. In every case the point P_4 traces a sphere C_4 such that C_1, C_2, C_3, C_4 are cut isogonally by C at P_1, P_2, P_3, P_4.

Theorems III and IV can be extended readily to any number of dimensions.

We add the remark that isogonal congruences arise in space naturally if we consider the problem of finding the locus of a point P_4 defined by the real constant cross ratio

$$\lambda = (P_1, P_2, P_3, P_4) \tag{3}$$

when the points P_1, P_2, P_3 have as their respective loci the regions R_1, R_2, R_3, or the surfaces S_1, S_2, S_3. The surfaces C_1, C_2, C_3 defining the isogonal congruence are the boundaries of the regions R_1, R_2, R_3 or the surfaces S_1, S_2, S_3 themselves. The boundary of the locus of P_4 is traced by P_4 as defined by (3) when the circle C of the congruence cuts C_1, C_2, C_3 isogonally at P_1, P_2, P_3. Detailed consideration of the corresponding fact for the plane has been given in a paper by the writer,* and can easily be extended to space by the reader.

HARVARD UNIVERSITY

A CORRECTION

BY EINAR HILLE

In the June number of this BULLETIN (vol. 28, No. 5, p. 261), the author published a paper with the title *Convex distribution of the zeros of Sturm-Liouville functions.* Through an oversight the last paragraph of the paper is inaccurate. We list the necessary corrections below.

Page 264, lines 4–10: Instead of "Note the lineal . . . φ_0.", read "On l we mark the eventual points a_n as well as the points where either arg $G(z) = \arg G(z_1) + \pi$ or $\theta_z \equiv \varphi_0 \pmod{\pi}$. Let $z_2 = z_2(\varphi_0)$ be the first of these points, different from z_1, which we encounter when proceeding along the ray, the rest of which we leave out."

Page 264, line 13: Instead of "an analytic curve", read "either of two analytic curves, namely $A(z_1)$ which is the locus arg $G(z) = \arg G(z_1) + \pi$, and".

Page 264, lines 24–27: Instead of "$l(\varphi_1)$ . . . respectively", read "$l(\varphi_1)$, considered as a double ray if necessary, from z_2^- to z_2 and from z_2 to z_2^+, we make the boundary curve continuous at $\varphi = \varphi_1$".

Page 264, line 28: Instead of "cuts", read "straight lines".

Page 264, line 30: After "the part of", insert "$A(z_1)$ and".

Page 265, first line: Leave out "on the cuts".

Same page, lines 6–10: Replace "Then . . . depends upon z_3" by "Then we can find an angle ϑ such that the two inequalities

$$(15)\qquad \begin{cases} \vartheta < \Theta < \vartheta + \pi; \\ 2k\pi < \Theta < 2(k+1)\pi; \end{cases} \qquad \Theta = \arg\,[G(z)(z - z_1)^2\,],$$

will hold for all interior points on the segment (z_1, z_3), where k is some integer".

PRINCETON UNIVERSITY

* See Lemma III of the paper to which reference has already been made, and also TRANSACTIONS OF THIS SOCIETY, vol. 23 (1922), pp. 67–88, Theorem II.