These conditions ensure the validity of the theorems* used in the proof of the existence theorem stated above, and the proof follows exactly as in the original theorem.

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ON THE CAUCHY-GOURSAT THEOREM.

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In order to prove his integral theorem, viz: $\int_C f(z)dz = 0$, Cauchy found it necessary to assume not only that the derivative $f'(z)$ existed but also that it was continuous. Later, proofs were given by Goursat and by Moore† in which the mere existence of $f'(z)$ was shown to be sufficient for the truth of the theorem. These were based upon the analysis of the complex variable.

From the standpoint of the real variable many interesting investigations have developed around the Cauchy-Goursat theorem. They have depended upon Green's theorem. Porter,‡ using the Riemann integral, proved that with proper restrictions upon the component functions, U and V, of the complex function, Green's theorem was true, and hence that Cauchy's integral theorem was also true, even when the derivative $f'(z)$ did not exist. Montel,§ by means of the Lebesgue integral, proved Green's theorem under the hypothesis that U_x, V_y, exist, are bounded, and satisfy the equation

$$U_x = V_y,$$

except at most in a set of measure zero. He was then able to prove the integral theorem, and the existence of the derivative $f'(z)$ for a function of the complex variable $f(z)$ in the closed region considered. The existence, then, of the deriva-

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* Cf. *Existence theorems for the general, real, self-adjoint linear system of the second order*, TRANSACTIONS AMER. MATH. SOC., vol. 19 (1918), p. 94.

† TRANSACTIONS AMER. MATH. SOCIETY, vol. 1 (1900).

‡ ANNALS OF MATHEMATICS, (2), vol. 7 (1905–6).

§ ANNALES SCIENTIFIQUES DE L'ECOLE NORMALE SUPÉRIEURE, (3), vol. 24 (1907).

tive is not necessary for the proof of the integral theorem under the restrictions that were imposed upon the partial derivatives by Montel or by Porter.

It is the purpose of this paper to show that Montel's restrictions as to the boundedness of the partial derivatives are not necessary. By means of the Denjoy integral* it will be shown that the line integral of the ordinary Green's theorem vanishes when taken along all closed curves that lie entirely inside the region and that possess a length. From this it follows that it is possible to establish the Cauchy integral theorem without assuming the existence of $f'(z)$.

We shall prove first the following theorem.

THEOREM. *If $U(x, y)$ and $V(x, y)$ are continuous functions of (x, y) which have finite partial derivatives, U_x, V_y, that satisfy the equation*

$$U_x = V_y$$

in each point of a closed region, R, then

$$(Dn)\int_b^{b'} (Dn)\int_a^{a'} U_x(x, y)dxdy,$$

$$(Dn)\int_a^{a'} (Dn)\int_b^{b'} U_x(x, y)dydx$$

exist and are equal for any rectangular region $a \leqq x \leqq a'$, $b \leqq y \leqq b'$ within R.

Since $U_x(x, y)$ is finite in R, we have

$$(1) \qquad (Dn)\int_a^{x} U_x(x, y)dx = U(x, y) - U(a, y).$$

Then

$$(Dn)\int_b^{y} (Dn)\int_a^{x} U_x(x, y)dxdy$$

$$= (Dn)\int_b^{y} [U(x, y) - U(a, y)]dy$$

exists, since $U(x, y)$ is continuous in (x, y). Similarly, since by hypothesis $U_x = V_y$, we can establish the existence of the integral

* The Denjoy integral will be denoted by the symbol $(Dn)\int$. Concerning its properties, see Denjoy, COMPTES RENDUS, vol. 154 (1912), pp. 859–862; and Hildebrandt, this BULLETIN, vol. 24 (1917), pp. 140–144.

$$(Dn)\int_a^{a'}(Dn)\int_b^{b'}U_x(x,y)dydx.$$

To prove the equality of these two double integrals we prove that

$$\frac{\partial}{\partial x}\frac{\partial F}{\partial y}=\frac{\partial}{\partial y}\frac{\partial F}{\partial x}, \tag{2}$$

where

$$F(x,y)=(Dn)\int_b^{y}(Dn)\int_a^{x}U_x(x,y)dxdy.$$

Since

$$F(x,y)=(Dn)\int_b^{y}[U(x,y)-U(a,y)]dy,$$

we have

$$\frac{\partial F}{\partial y}=U(x,y)-U(a,y),$$

whence

$$\frac{\partial}{\partial x}\frac{\partial F}{\partial y}=U_x(x,y)=V_y(x,y). \tag{3}$$

Differentiating $F(x,y)$ first with respect to x, we find

$$\frac{\partial F}{\partial x}=\operatorname*{Lim}_{\Delta x=0}\frac{\Delta F}{\Delta x}=\operatorname*{Lim}_{\Delta x=0}(Dn)\int_b^{y}\frac{U(x+\Delta x,y)-U(x,y)}{\Delta x}dy$$

$$=\operatorname*{Lim}_{\Delta x=0}(Dn)\int_b^{y}U_x(x+\theta\Delta x,y)dy.$$

Since $U_x=V_y$, we have

$$\frac{\partial F}{\partial x}=\operatorname*{Lim}_{\Delta x=0}(Dn)\int_b^{y}U_x(x+\theta\Delta x,y)dy$$

$$=\operatorname*{Lim}_{\Delta x=0}(Dn)\int_b^{y}V_y(x+\theta\Delta x,y)dy$$

$$=\operatorname*{Lim}_{\Delta x=0}[V(x+\theta\Delta x,y)-V(x+\theta\Delta x,b)]$$

$$=V(x,y)-V(x,b).$$

It follows that

$$\frac{\partial}{\partial y}\frac{\partial F}{\partial x}=V_y(x,y)=U_x(x,y),$$

whence, by (3),

$$\frac{\partial}{\partial x}\frac{\partial F}{\partial y} = \frac{\partial}{\partial y}\frac{\partial F}{\partial x}. \tag{4}$$

Integrating the left member of (4) with respect to x, we have

$$(Dn)\int_a^{a'} \frac{\partial}{\partial x}\frac{\partial F}{\partial y}dx = \frac{\partial F}{\partial y}(a', y) - \frac{\partial F}{\partial y}(a, y)$$
$$= (Dn)\int_a^{a'} U_x(x, y)dx,$$

whence

$$(Dn)\int_b^{b'}(Dn)\int_a^{a'}\frac{\partial}{\partial x}\frac{\partial F}{\partial y} = (Dn)\int_b^{b'}\left[\frac{\partial F}{\partial y}(a', y) - \frac{\partial F}{\partial y}(a, y)\right]dy \tag{5}$$
$$= (Dn)\int_b^{b'}(Dn)\int_a^{a'} U_x(x, y)dxdy$$
$$= F(a', b') - F(a', b) - F(a, b') + F(a, b).$$

Integrating the right member of (4) with respect to y and then with respect to x, we get

$$(Dn)\int_a^{a'}(Dn)\int_b^{b'}\frac{\partial}{\partial y}\frac{\partial F}{\partial x}dydx$$
$$= F(a', b') - F(a', b) - F(a, b') + F(a, b),$$

whence

$$(Dn)\int_b^{b'}(Dn)\int_a^{a'}\frac{\partial}{\partial x}\frac{\partial F}{\partial y}dxdy = (Dn)\int_a^{a'}(Dn)\int_b^{b'}\frac{\partial}{\partial y}\frac{\partial F}{\partial x}dydx$$

or

$$(Dn)\int_b^{b'}(Dn)\int_a^{a'} U_x(x, y)dxdy \tag{6}$$
$$= (Dn)\int_a^{a'}(Dn)\int_b^{b'} U_x(x, y)dydx.$$

To prove $\int_C Udy + Vdx = 0$, where C is a closed curve that has a length, we consider first the case in which C is a rectangle. The application of this theorem to the line integral over the rectangle follows from (6) and the equality $U_x = V_y$. We have

$$\int_C Udy + Vdx = 0,$$

where the Denjoy integral reduces to the Riemann integral, since U and V are continuous.

By well known methods* this can be extended to any rectifiable curve bounding a simply connected region. Hence we have the following theorem.

THEOREM. *If U and V are continuous functions of (x, y) possessing finite partial derivatives U_x, V_y, and if*

$$U_x = V_y$$

in a region R, then

$$\int_C Udy + Vdx = 0$$

over any rectifiable curve C bounding a simply connected domain lying wholly within R.

The proof of the Cauchy integral theorem is immediate. Moreover if this theorem is satisfied by a continuous function $f(z)$, we know that $f(z)$ is analytic. We may then state our results as follows.

THEOREM. *The necessary and sufficient conditions that $\int f(z)dz = 0$ over any rectifiable curve C bounding a simply connected region lying entirely within a domain R are that the component functions U, V of $f(z) = U + iV$ possess finite partial derivatives in R and that they satisfy the Cauchy-Riemann equations*

$$U_x = V_y, \quad U_y = -V_x.$$

It may be noted that the existence† of $f'(z)$ in any point z_0 requires:

(*a*) The existence of U_x, U_y, V_x, V_y;

(*b*) $U_x = V_y$; $U_y = -V_x$;

(*c*) $\Delta U - \left(\frac{\partial U}{\partial x}\Delta x + \frac{\partial U}{\partial y}\Delta\right)$, $\Delta V - \left(\frac{\partial V}{\partial x}\Delta x + \frac{\partial V}{\partial y}\Delta y\right)$

infinitely small with respect to $|\Delta x| + |\Delta y|$. Hence the hypotheses of the preceding theorem are less restrictive than those of Goursat.‡

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* Goursat, *Mathematical Analysis*, vol. II, part 1, p. 67.

† Fréchet, NOUVELLES ANNALES, vol. 19 (1919), p. 219.

‡ TRANSACTIONS, loc. cit.