THE CHICAGO MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

The fifteenth regular Western Meeting of the American Mathematical Society was held at the University of Chicago on Wednesday and Thursday, December 29 and 30, 1920. This meeting was the forty-sixth regular meeting of the Chicago Section, and was held in affiliation with the Convocation week meetings of the American Association for the Advancement of Science.

The total attendance was more than one hundred, including the following eighty-three members of the Society: Professor R. P. Baker, Professor A. A. Bennett, Professor Henry Blumberg, Professor P. P. Boyd, Professor W. D. Cairns, Professor Florian Cajori, Professor J. A. Caparo, Professor R. D. Carmichael, Professor E. W. Chittenden, Professor C. E .Comstock, Professor H. H. Conwell, Professor D. R. Curtiss, Professor W. W. Denton, Professor L. E. Dickson, Professor Arnold Dresden, Professor L. C. Emmons, Professor H. J. Ettlinger, Professor G. C. Evans, Professor T. M. Focke, Professor G. H. Graves, Professor W. L. Hart, Professor E. R. Hedrick, Professor T. H. Hildebrandt, Professor T. F. Holgate, Professor Dunham Jackson, Professor O. D. Kellogg, Professor A. M. Kenyon, Professor E. P. Lane, Professor A. O. Leuschner, Mrs. M. I. Logsdon, Professor Gertrude I. McCain, Professor W. D. MacMillan, Professor H. W. March, Professor William Marshall, Professor T. E. Mason, Professor L. C. Mathewson, Professor E. D. Meacham, Professor G. A. Miller, Professor W. L. Miser, Professor U. G. Mitchell, Professor C. N. Moore, Professor E. H. Moore, Professor E. J. Moulton, Professor F. R. Moulton, Dr. J. R. Musselman, Professor G. W. Myers, Dr. C. A. Nelson, Mr. H. L. Olson, Professor C. I. Palmer, Professor A. D. Pitcher, Professor L. C. Plant, Professor S. E. Rasor, Professor J. F. Reilly, Professor H. L. Rietz, Professor W. J. Risley, Professor Maria M. Roberts, Professor W. H. Roever, Professor D. A. Rothrock, Dr. A. R. Schweitzer, Professor G. T. Sellew, Professor W. G. Simon, Professor E. B. Skinner, Professor D. E. Smith, Professor E. R. Smith, Professor G. W. Smith, Professor I. W. Smith, Dr. L. L. Steimley, Professor R. B. Stone, Professor E. B.

Stouffer, Professor E. J. Townsend, Dr. B. M. Turner, Professor E. B. Van Vleck, Professor Warren Weaver, Professor W. P. Webber, Mr. F. M. Weida, Professor W. D. A. Westfall, Professor E. J. Wilczynski, Professor C. E. Wilder, Professor D. T. Wilson, Professor R. E. Wilson, Professor C. H. Yeaton, Professor J. W. Young, Professor W. A. Zehring.

The session of Wednesday forenoon was a joint meeting with the Mathematical Association of America and with Sections A and L of the American Association for the Advancement of Science. At this meeting, which was presided over by Professor D. R. Curtiss, chairman of Section A, Professor O. D. Kellogg gave his address entitled *A decade of American mathematics,* as retiring vice-president of Section A of the American Association for the Advancement of Science. His paper was followed by an illustrated paper on *The evolution of algebraic notations,* by Professor Florian Cajori.

The meeting of Wednesday afternoon was held simultaneously with a meeting of the Mathematical Association of America, and was presided over by the chairman of the Chicago Section, Professor R. D. Carmichael. During the Thursday sessions, Professor Carmichael was relieved in the chair by Professor Dunham Jackson, newly-elected vice-president of the Society, and by Professor G. A. Miller.

On Wednesday evening a joint dinner of the Chicago Section, the Mathematical Association of America, the American Astronomical Society, and Sections A and D of the American Association for the Advancement of Science, was held at the Quadrangle Club. At this dinner about one hundred seventy-five persons were present. Professor D. E. Smith, retiring president of the Mathematical Association of America, acted as toastmaster. Short speeches were made by representatives of the different participating organizations. An interesting feature of this dinner consisted of the exhibition by Professor C. I. Palmer of a copy of the 1637 edition of Descartes' Geometry, to which Professor Cajori had alluded in the paper referred to above.

At this meeting the following papers were read:

1. Professor W. P. Webber: *Construction of doubly periodic functions with singular points in the period parallelogram.*

Professor Webber starts with the Weierstrassian p-function and arrives at the function

$$W_2(u) = e^{p(u)} - 1,$$

which is doubly periodic, has zeros at the zeros of $p(u)$ and essential singular points at the poles of $p(u)$. This function is used for constructing more general ones.

It is shown that $W_2(u)$ cannot have an algebraic addition theorem. Among others, the following relations are established: $W_2(u) = [W_2(u) + 1]R(p(u), p'(u))$, where R denotes a rational integral function; $W_2(u) + 1 = e^{F_1(W_2(u), W_2'(u))}$, where F_1 is one of the roots of the cubic in $p(u)$, $4\{p^3 - g_2p - g_3\} = \{W_2'(u)/[W_2(u) + 1]\}^2$, g_2, g_3, being the Weierstrassian invariants for $W_2(u)$. Series are derived for $W_2(u)$.

2. Professor H. J. Ettlinger: *Boundary value problems with regular singular points. Second paper.*

In this paper Professor Ettlinger continues his investigation of the oscillatory properties of the solutions of a second order linear boundary value problem, having regular singular points at the end points of the interval.

3. Professor E. W. Chittenden: *Note on the permutability of functions which have the same Schmidt fundamental functions.*

Let $\varphi_i(x)$, $\psi_i(x)$ denote the normalized Schmidt fundamental functions of a kernel $K(x, y) = \Sigma_{i=1}^{\infty} k_i\varphi_i(x)\psi_i(y)$. Professor Chittenden determines the most general function of the form $H(x, y) = \Sigma_{i=1}^{\infty} h_i\varphi_i(x)\psi_i(y)$ which is permutable of the second kind with $K(x, y)$.

4. Professor E. W. Chittenden: *On kernels which have no Fredholm fundamental functions.*

Professor Chittenden studies the classes of kernels which have no Fredholm fundamental functions. For example, if $\varphi_n(x)$, $\psi_n(x)$, $(n = 0, \pm 1, \pm 2, \cdots)$ is a closed normalized biorthogonal system of functions on an interval $(a \leqq x \leqq b)$, the function $K(x, y) = \Sigma_{n=-\infty}^{+\infty} a_n\varphi_n(x)\psi_{n+p}(y)$ (assuming convergence in mean) where the a_n are all different from zero and p is a positive integer, has no Fredholm fundamental functions, and is not orthogonal on the right or left to any function φ such that $\int_a^b \varphi^2 dx > 0$. In case $a_n = 1/n^2$ and the functions φ_n, ψ_n are continuous, the function $K(x, y)$ is continuous.

5. Professor E. W. Chittenden: *Note on convergence in the mean.*

Professor Chittenden presents an example of a sequence of functions which converges in the mean on an interval ($a \leqq x \leqq b$), but diverges at every point of the interval.

6. Dr. A. R. Schweitzer: *Determination of the spherical transformation in Grassmann's extensive algebra.*

The essence of the geometric application of the quaternions in Hamilton's analysis is to be found in the derivation of Euler's transformation. The attempt of Gibbs to derive these equations by means of Grassmann's Lückenausdrücke appears at a disadvantage when compared with Hamilton's simple deduction based on the transformation qvq^{-1}. Using his article in the MATHEMATISCHE ANNALEN, vol. 69, as a basis, Dr. Schweitzer gives a new derivation of Grassmann's circular transformation and by direct generalization determines the spherical transformation. In this derivation, the simplicity of Hamilton's deduction is completely preserved; it is based on the definitions $X \cdot Q(E_iE_j) = x_1E_1Q(E_iE_j) + x_2E_2Q(E_iE_j) + x_3E_3Q(E_iE_j) + x_4E_4Q(E_iE_j)$, $E_kQ(E_iE_j) = Q(E_kE_i) \cdot E_j$, where $i, j, k = 1, 2, 3, 4$, and where the notation is the same as in the article cited above.

7. Dr. A. R. Schweitzer: *On the relation of iterative compositional equations to Lie's theory of transformation groups.*

The theory of Lie's transformation groups may be interpreted as a theory on the solution of certain functional equations of the iterative compositional type, subject to auxiliary conditions. On this basis, the group property arises through the introduction of a suitable notation. In the MATHEMATISCHE ANNALEN, vol. 18, Lie mentions that his one-dimensional, r-parameter functional equations are a generalization of the equation $\phi\{\phi(x, a), b\} = \phi\{x, \phi(a, b)\}$, which Lie seems wrongly to ascribe to Abel, but which is apparently due to C. J. Hill. Dr. Schweitzer shows that a new definition of transformation groups arises by generalizing the functional equation

$$f\{f(x, a), f(b, a)\} = f(x, b)$$

and that certain of his *quasi-distributive* equations are a degenerate case of Lie's n-dimensional, n-parameter functional equations and that certain of his quasi-transitive functional equations are a degenerate case of the quasi-transitive correlative of Lie's n-parameter, n-dimensional functional equations. On the other hand, Dr. Schweitzer's quasi-transitive equations and their inverse correlatives suggest generalizations of Lie's functional equations and his concept transformation group. The memoirs of Schur in the MATHEMATISCHE ANNALEN treating the analytic and non-analytic solutions of Lie's functional equations readily give direction to the research in the analytic and non-analytic solutions of equations in iterative compositions in general.

8. Professor E. J. Wilczynski: *Isothermally conjugate nets.*

The geometric properties which characterize an isothermally conjugate net were, until recently, entirely unknown. One very elegant characterization was given by the late Dr. G. M. Green. But Green overlooked a case in which his criterion does not distinguish between isothermally conjugate nets and other nets of an entirely different character. In this paper, which has appeared in the AMERICAN JOURNAL OF MATHEMATICS, October, 1920, Professor Wilczynski shows how to complete Green's discussion by introducing an important new concept, namely, that of a *pencil* of conjugate nets, at least in the special case of *isothermally* conjugate nets.

9. Professor E. J. Wilczynski: *Transformation of conjugate nets into conjugate nets.*

In this second paper, Professor Wilczynski shows how the projective invariants and covariants of a conjugate net are affected by transformations which change it into a new conjugate net on the same surface. The notion of a pencil of conjugate nets is here developed in its general form, enabling him to simplify in a notable fashion the results of the preceding paper. But there result, at the same time, two other characteristic properties of isothermally conjugate systems which are entirely different in kind from Green's criterion.

10. Professor R. L. Moore: *Conditions under which one of two given closed linear point sets may be thrown into the other one by a continuous transformation of a plane into itself.*

It is easy to show, by the exhibition of examples, that if S_n is a space of one or more dimensions, there exist in S_n two closed, bounded point sets which are in one-to-one continuous correspondence with each other but neither of which can be thrown into the other by a continuous one-to-one transformation of S_n into itself. Professor Moore raises the question whether, if S_n is an n-dimensional euclidean space lying in an $n + 1$ dimensional euclidean space S_{n+1}, J_1 and J_2 are closed, bounded point sets lying in S_n, and there is a continuous one-to-one correspondence between the points of J_1 and the points of J_2, there exists a continuous one-to-one transformation of S_{n+1} into itself which throws J_1 into J_2. He shows that in the case where $n = 1$ this question can be answered in the affirmative.

11. Professor R. L. Moore: *A closed connected set of points which contains no simple continuous arc.*

Some time ago R. L. Moore and J. R. Kline proposed the following questions: (1) Does there exist a non-degenerate* closed connected set of points which contains no simple continuous arc? (2) Does there exist a non-degenerate connected set of points which contains no arc? In a paper presented to this Society in December, 1919, G. A. Pfeiffer gave an example† showing that the second question may be answered in the affirmative. It is to be noted, however, that not only does the set of points exhibited in this example fail to be closed but it does not even contain a single non-degenerate connected subset that is closed. It, therefore, does not furnish an answer to the first question. In the present paper Professor Moore shows that the first question also may be answered in the affirmative.

12. Professor Florian Cajori: *On the history of symbols for n-factorial.*

Professor Cajori points out the origin of the symbol $\underline{|n}$ for n-factorial and traces the spread of this symbol and of $n!$, particularly in the United States. The paper will be published in ISIS.

13. Professor L. E. Dickson: *Homogeneous polynomials with a multiplication theorem.*

In his first paper, Professor Dickson investigates all homogeneous polynomials $f(x_1, \cdots, x_n) \equiv f(x)$ of degree d such that $f(x)f(\xi) \equiv f(X)$, where $X_1, \cdots, X_n$ are bilinear functions

* A set of points is said to be non-degenerate if it contains more than one point.

† Cf. this BULLETIN, vol. XXVI (1920), p. 246.

of $x_1, \cdots, x_n$ and $\xi_1, \cdots, \xi_n$. Known examples of f are determinants whose ($n = r^2$) elements are independent variables, norms of algebraic numbers, and sums of 2, 4, or 8 squares. It is convenient to introduce the linear algebra in which the product of $x = \Sigma x_i e_i$ by $\xi = \Sigma \xi_i e_i$ is $X = \Sigma X_i e_i$. It is proved that $f(x)$ must divide the dth powers of the determinants $\Delta(x)$ and $\Delta'(x)$ of the general number of the algebra. It is assumed henceforth that $f(x)$ is not expressible in fewer than n variables. By means of a linear transformation on the x's which leaves $f(x)$ unaltered and one on the ξ's leaving $f(\xi)$ unaltered, we secure the simplification that the new composition $x\xi = X$ takes place in a linear algebra having a principal unit e_1 such that $e_1 x = x e_1 = x$ for every x in the algebra. Now every factor of $f(x)$ admits the composition. Next, if $f(x)$ has a covariant of degree δ and index λ which is not identically zero, then f^δ is divisible by Δ^λ and Δ'^λ. Hence, if $\lambda > 0$, f has the same irreducible factors as Δ and Δ', which also admit the composition. For $n < 5$, the hypothesis is satisfied since the Hessian of f is not identically zero (Gordan and Nöther, MATHEMATISCHE ANNALEN, vol. 10, 1876, p. 564). Further, any covariant of f is then a power of f. But an irreducible quartic surface $f = 0$ whose Hessian is cf^2 is a developable surface whose edge of regression is a twisted cubic curve. By means of these theorems it is shown that, if $n < 5$, f is a product of powers of n linear forms or a power of a quaternary quadratic form. For $n = 5$, the former theorems hold, but the case of an irreducible Δ remains undecided. The paper will appear in the PROCEEDINGS OF THE INTERNATIONAL CONGRESS AT STRASBOURG.

14. Professor L. E. Dickson: *Applications of algebraic and hypercomplex numbers to the complete solution in integers of quadratic diophantine equations in several variables.*

In this second paper, Professor Dickson has obtained formulas giving the complete solution in integers of certain equations, as $x_1^2 + x_2^2 + x_3^2 = x_4^2$, $x_1^2 + \cdots + x_5^2 = x_6^2$, when the parameters take only integral values. Transposing one square, we have to express a product as a sum of 2 or 4 squares. Evident solutions are furnished by the theorem on the norm of a product of two complex integers or two quaternions. When the numbers in the resulting formulas are multiplied by the same number ρ and when ρ is allowed to take all integral values, the products are shown to give all integral solutions. Use is made of the fact that complex integers obey the laws of arithmetic, and likewise quaternions with integral coordinates provided one of the quaternions used has an odd norm. A like theory applies to $x^2 + y^2 + kz^2 = w^2$, when $k = \pm 2, \pm 3, -5, 7, 11, -13$. But if $k = 5$, the numbers $x + y\sqrt{-5}$ do not obey the laws of arithmetic. Corresponding to the two classes of ideals, there are two distinct sets of formulas which together give all integral solutions. This application of ideals will be developed in a later paper. The remaining topics were presented in detail before the International Congress at Strasbourg and will appear in the PROCEEDINGS OF THE CONGRESS.

15. Professor L. E. Dickson: *Arithmetic of quaternions.*

In this third paper, Professor Dickson recalled that A. Hurwitz (GÖTTINGER NACHRICHTEN, 1896, p. 313) proved that the laws of arithmetic hold for integral quaternions, viz. those whose coordinates are either all integers or all halves of odd integers. Since fractions introduce an inconvenience in applications to Diophantine analysis, it is here proposed to define an integral quaternion to be one whose coordinates are all integers. It is called odd if its norm is odd. It is proved that, if at least one of two integral quaternions a and b is odd, they have a right-hand greatest common divisor d which is uniquely determined up to a unit factor ($\pm 1, \pm i, \pm j, \pm k$), and that integral quaternions A and B can be found such that $d = Aa + Bb$. Similarly there is a left-hand greatest common divisor expressible in the form $a\alpha + b\beta$. The further theory proceeds essentially as in Hurwitz's exposition. The paper has been offered to the PROCEEDINGS OF THE LONDON MATHEMATICAL SOCIETY.

16. Professor L. E. Dickson: *Determination of all general homogeneous polynomials expressible as determinants with linear elements.*

In the fourth paper by Professor Dickson it is proved that every binary form, every ternary form, every quaternary quadratic form, and a sufficiently general quaternary cubic form can be expressed as a determinant whose elements are linear forms, while no further general form has this property. For a plane curve $f = 0$ of order r, we may assume without loss of generality that f has no repeated factor. Then some line cuts the curve in r distinct points; take it as the side $z = 0$ of a triangle of reference. As the side $y = 0$, take any line not meeting $z = 0$ at one of its r intersections with the curve. Then for $z = 0$, f is a product of r distinct linear functions $X_i = x + \lambda_i y$. It is proved that f can be expressed in one and but one way as a determinant

$$\begin{vmatrix} X_1 + c_{11}z & z & 0 & \cdots & 0 \\ c_{21}z & X_2 + c_{22}z & z & \cdots & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot \\ c_{r1}z & c_{r2}z & c_{r3}z & \cdots & X_r + c_{rr}z \end{vmatrix},$$

in which the elements above the diagonal are zero except for the elements z just above it. The paper will appear in the TRANSACTIONS. A related paper, dealing with quaternary cubic forms with attention to rationality, has been offered to the AMERICAN JOURNAL OF MATHEMATICS.

17. Professor G. A. Miller: *I-conjugate operators of an abelian group.*

Two operators of any group G are said to be *I-conjugate* if they correspond in at least one of the possible automorphisms of G. As G is supposed to be abelian, it is possible to find a set of generators of $G(s_1, s_2, \cdots, s_\rho)$ such that the group generated by any arbitrary subset of these ρ operators has only the identity in common with the group generated by the rest of them. Any operator of G expressed in terms of these generators is called *I-reduced* if the number of its constituents which are powers of these operators is as small as possible for the set of I-conjugates to which it belongs. Professor Miller then notes that a necessary and sufficient condition that every I-reduced operator of G involve only one constituent is that the quotient of no two invariants of G exceed p when the order of G is p^m, p being a prime number. He also proves that the smallest groups which involve characteristic operators of odd order are two groups of order 63 and that a necessary and sufficient condition that the quotient groups corresponding to the subgroups generated by two I-reduced operators be of the same type, is that the constituents of these operators generate the same subgroups.

18. Professor W. D. Macmillan: *The integrals*

$$\int_0^x e^{x^2/2}dx, \quad \int_0^x \sin\frac{x^2}{2}\,dx, \quad \int_0^x \cos\frac{x^2}{2}\,dx,$$

and associated divergent series.

It is evident that the integral $\int_0^x e^{x^2/2}dx$ cannot be completely tabulated and thus brought into the category of known functions, because it increases beyond all limits and it has no simple properties which enable us to determine its value for a given large value of x. But if we write $\int_0^x e^{x^2/2}dx = e^{x^2/2}\cdot\varphi(x)$, then $\varphi(x)$, for large values of x, is a decreasing function which has the limit zero. It admits the divergent expansion

$$\varphi(x) \equiv \frac{1}{x} + \frac{1}{x^3} + \frac{1\cdot 3}{x^5} + \frac{1\cdot 3\cdot 5}{x^7} + \cdots,$$

which is very useful for computing the numerical value of φ for large values of x. It is shown by Professor MacMillan that if T_n is the nth term of this series and $\varphi_{n-1}(x)$ is the sum of the first $n - 1$ terms of $\varphi(x)$, and if $2n - 3 < x^2 < 2n - 1$, then T_n is the minimum term and $|\varphi(x) - \varphi_{n-1}(x)| < T_n(x)$; i.e. the error committed in using the divergent series is less than the minimum term provided the series is carried up to, but does not include, the minimum term. Thus, for $x \geqq 6$, the series will give results certainly accurate to 8 decimals, and the order of accuracy increases rapidly with x. A table of values for x up to 6 is given to 7 decimal places. Similar remarks apply to the other integrals.

19. Professor R. P. Baker: *Elementary geometry in n dimensions.*

In this paper, Professor Baker provides a standard method for finding distance and angles between flat spaces in n-dimensional euclidean geometry. Each R_k is supposed given by a matrix $(k + 1)(n + 1)$ of rectangular coordinates with a column of units. The joint invariants of a pair of such flat spaces are (1) a *mutual moment* involving the distance and all the angles, (2) an *inner product* involving all the angles, and (3) a set of *extensionals* of the general type of mutual moments of figures simply constructed from the data. These furnish symmetric functions of the *slopes*. The algebraic equation for determining the slopes always has real roots.

Since the mutual moment of two lines in space is a polarized form of Plücker's identity, the extensionals are polarized forms of identically vanishing relations (extensionals of Plücker's, etc.) of the determinants of the space. The difficulty of dealing with supernumerary coordinates is surmounted by a wide extension of the theorem of Pythagoras. The *content* of any $(k + 1)$ point in R_n is the square root of the sum of the squares of the projections on the axial R_k's. This applies directly to the mutual moment and extensionals and also to the inner product when regression is used to reduce to the case $(k, k, 2k)$. The theory is linear algebra, though regression is used as a device. If progressive methods are desired in connection with the general Pythagorean theorem, we can construct the square of two flat $(k + 1)$ points, which is equivalent to the sum of the square of the projections. Models of an R_3 perspective and development of the *square on the triangle* are shown. Finite dissection, however, fails for tetrahedra, Dehn's condition not being satisfied in general.

20. Professor Dunham Jackson: *Note on an ambiguous case of approximation.*

In recent papers, Professor Jackson has discussed the existence and properties of a trigonometric sum $T_{mn}(x)$ of order n at most, determined by the condition that it shall give the best possible approximation to a given continuous periodic function $f(x)$, in the sense of the integral of the mth power of the absolute value of the error. In these discussions it has been assumed that $m \geqq 1$. The purpose of the present note is to inquire what becomes of the problem for values of m less than 1. It is immediately seen that the problem degenerates for $m \leqq 0$, so that there is occasion to consider only values of m between 0 and 1. It is found that there is always at least one determination of $T_{mn}(x)$ which makes the integral a minimum, but that this determination is not generally unique. Nevertheless, it is possible to treat the question of convergence, when m is held fast and n is allowed to become infinite, in substantially the same way as for $m \geqq 1$. In case there are two or more determinations of $T_{mn}(x)$ for a given value of n, it is immaterial for the convergence which is chosen. The treatment applies equally well to the problem of polynomial approximation, and is in part of still wider application.

21. Professor Dunham Jackson: *On the method of least mth powers for a set of simultaneous equations.*

If there is given a set of p simultaneous linear equations in n unknown quantities, $p > n$, the question may be raised of determining values for the unknowns so that the equations shall be approximately solved, in the sense that the sum of the mth powers of the absolute values of the errors is a minimum. For $m = 2$, this is the classical problem of least squares. Professor Jackson treats the general problem by methods analogous to those used in recent papers on the approximate representation of a function of a continuous variable x, the independent variable now being represented by an index ranging from 1 to p. While there is a general correspondence between the two cases, continuous and discrete, both as to methods and as to results, the parallelism in detail does not seem to be so close as to render a separate treatment of the algebraic problem superfluous.

22. Professor Dunham Jackson: *Note on the convergence of weighted trigonometric series.*

In this note, Professor Jackson shows that a method used recently in connection with certain problems of convergence in the theory of trigonometric and polynomial approximations can be applied to a more general class of problems, including some cases considered on the formal side by Gram (Crelle, vol. 94) in which different weights are assigned to different values of the independent variable.

23. Professor A. J. Kempner: *On polynomials and their residue systems. Second paper.*

In this paper Professor Kempner continues and develops his earlier paper, *On polynomials and their residue systems,* read at the Chicago meeting of the Society in December, 1917. Residue systems of polynomials with respect to a composite numerical modulus are systematically examined.

24. Professor E. R. Smith: *Expansion of the double-frequency function into a series of Hermite's polynomials.*

Dr. Smith gives a development of the general double-frequency function into a series of polynomials which were first investigated by Hermite. The result may be expressed in terms of the successive partial derivatives of the normal double-frequency function. It is shown that the results are applicable when the regression is non-linear. The generalized form of the equation defining the curve of regression is also discussed.

25. Professor T. E. Mason: *On amicable numbers and their generalizations.*

In this paper Professor Mason presents some new pairs of amicable numbers and some generalized amicable number sets. Dickson defined an amicable k-tuple as k numbers n_1, $n_2, \cdots, n_k$ satisfying the equations

$$S(n_1) = S(n_2) = \cdots = S(n_k) = n_1 + n_2 + \cdots + n_k,$$

where $S(n)$ means the sum of all the divisors of n. Amicable k-tuples are given for $k = 3, 4, 5, 6$. Carmichael defined multiply amicable numbers as numbers m and n satisfying the equations $S(m) = S(n) = t(m + n)$. Multiply amicable number pairs are given for $t = 2, 3$, and multiply amicable triples for $t = 2$.

26. Professor Henry Blumberg: *On the complete characterization of the set of points of approximate continuity.*

It is known that the set C of points where a function is continuous is a ΠI_n, i.e. a product of a denumerable sequence of sets I_n, each of which consists exclusively of inner points; conversely, if a set S is a ΠI_n, then a function exists which is continuous at every point of S and discontinuous elsewhere. If we think of continuity as equivalent to the vanishing of the ordinary saltus (or oscillation, or least upper bound minus greatest lower bound), we are led to the notion of *approximate* continuity of different kinds when, in place of the ordinary saltus, we employ the f-saltus, the d-saltus, the z-saltus, etc., which are obtained by regarding as negligible finite sets, denumerable sets, sets of zero measure, etc., respectively. Hence, we are led to seek a characterization of the sets C_f, C_d, C_z, etc., of points of approximate continuity. Professor Blumberg deals with these and related questions. One of the results is that the sets C_f, C_d, C_z are completely characterized as being expressible in the form ΠI_n.

27. Dr. Gladys E. C. Gibbens: *Comparison of different line-geometric representations for functions of a complex variable.*

In the present paper, Dr. Gibbens generalizes the methods given by Professor Wilczynski (TRANSACTIONS, vol. 20) for constructing a rectilinear congruence by means of a functional relation between two complex variables. She finds that as long as the planes of the two complex variables remain parallel, the projective properties of the class of congruences defined by means of the totality of all analytic functions $w = F(z)$ are independent of the relative position of the origins, of the angle between the real axes of the two complex variables, and of the distance between the planes. The congruence which corresponds to an individual function $w = F(z)$ in any particular representation corresponds not to itself, but to the function $e^{i\theta}w = F(z)$ if the angle between the real axes of the two planes be changed by θ.

If the planes of the two complex variables are not parallel, the congruence defined by the particular functional relation $w = F(z)$ is projectively equivalent to that defined by the function $e^{i\theta_2}w = F(e^{i\theta_1}z)$, where the planes of the new variables $W = e^{i\theta_2}w$; $Z = e^{i\theta_1}z$ are perpendicular to each other, and where the real axes have been rotated through angles θ_1, θ_2, respectively, in order to become parallel to the line of intersection of the planes. The reality of the focal sheets and the developables of the congruence depends upon the particular function under consideration.

If the two complex variables are projected from a common plane upon concentric spheres, and corresponding points joined, the properties of the resulting congruence are again dependent upon the particular functional relation assumed between the two complex variables. In the last two cases, the generalizations are, so far, inadequate from the point of view of a general theory.

28. Professor Dunham Jackson: *On the trigonometric representation of an ill-defined function.*

The ordinary notion of single-valued function of a variable x can be generalized in the following manner: It may be supposed that the value of y corresponding to any given x is indeterminate, and that any value, within bounds, is to be regarded as possible, but that some values are more probable than others, to an extent indicated by a weight which is a function of x and y, essentially positive or zero, That is, such a function of two variables, $\omega(x, y)$, can be regarded as constituting a sort of function of the single variable x, which is blurred or out of focus, and can be made distinct only by a more or less arbitrary averaging process. Professor Jackson develops the idea thus vaguely expressed, by a study of certain properties of non-negative functions $\omega(x, y)$. It is shown how the method of least squares, or, more generally, of least mth powers, can be used to determine a definite trigonometric sum, of any prescribed order, which may be regarded as representative of the given indeterminate function, supposed periodic with regard to x; and it is proved that, under fairly general hypotheses, the trigonometric sum thus obtained converges uniformly to a suitably defined single-valued average.

29. Professor E. L. Dodd: *An adaptation of Bing's paradox, involving an arbitrary a priori probability.*

Professor Dodd finds that the essential feature of Bing's paradox remains, even after the justly criticized* constant a priori probability is replaced by an arbitrary non-negative continuous probability $w(x)$. If s men of given age all live a moment, the probability that another man of that age will live a moment is $p = \int_0^1 w(x)x^{s+1}dx / \int_0^1 w(x)x^s dx$.

If the s men all live a year of n moments then the probability that the other man also will live a year is p^n, and this approaches zero with increasing n since $p < 1$. It is practically certain therefore, that the other man will die within the year. It should be noted, however, that the use of the same $w(x)$ for a moment of sixty seconds and for a moment of one second can hardly be justified. Thus n, though it may be large, is not subject to indefinite increase.

30. Professor O. D. Kellogg: *A convergence theorem and an application.*

In this note, Professor Kellogg gives a simple proof of a theorem due to Ascoli on the uniform convergence of a subsequence of equi-continuous functions and makes an application to the proof of the existence of a characteristic function for a symmetric kernel.

31. Professors G. D. Birkhoff and O. D. Kellogg: *Invariant points under transformations in function space.*

If a closed convex region of the plane be mapped into itself by a continuous single-valued transformation, there will be

* Arne Fisher, *The Mathematical Theory of Probabilities*, p. 75, quoting Kroman.

an invariant point. Professors Birkhoff and Kellogg propose to extend this theorem to n dimensions and to the space of continuous functions. Certain existence theorems may be formulated in terms of the invariance of a point in a function space under a transformation, and it is proposed to consider such applications of the theorem.

32. Professor G. C. Evans: *Fundamental points of potential theory.*

Professor Evans treats some of the problems of harmonic functions and monogenic functions by means of Lebesgue-Stieltjes integrals and generalized derivatives.

33. Professor W. L. Hart: *Functionals of summable functions.*

Let H represent the class of all functions $u(x)$ on an interval $a \leqq x \leqq b$ which, together with their squares, are summable in the Lebesgue sense on (a, b). In the first part of his paper, Professor Hart considers real-valued functionals $F[u]$ defined for all u in H. It is assumed that if a sequence u_n of H converges in the mean to a function u, then $\lim_{n=\infty} F[u_n] = F[u]$. A representation of $F[u]$ in an infinite series is obtained; a mean-value theorem is obtained for the case that $F[u]$ has a differential; an infinite system of functional equations is solved for an unknown function $u(x, s)$, s being a parameter. In the second part of the paper there are considered functionals $G[u; t]$ which for every u of H yield functions of t that belong to H for $c \leqq t \leqq d$. The concepts continuity and differential are defined. The Fourier coefficients of $G[u; t]$ with respect to a complete orthogonal system $[\varphi_1(t), \varphi_2(t), \cdots]$ are found to be functionals of the type $F[u]$. There are obtained for $G[u; t]$ an infinite series converging in the mean, a mean-value theorem, the solution of implicit functional equations, and a treatment of a pseudo-differential functional equation. In many proofs in the paper, fundamental use is made of theorems and definitions relating to functions of infinitely many variables and to pseudo-derivatives, previously treated by the author.

The papers of Professors Dodd, G. C. Evans, and R. L. Moore, and the paper of Dr. Gibbens, were read by title.

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