SHEFFER'S SET OF FIVE POSTULATES FOR BOOLEAN ALGEBRAS IN TERMS OF THE OPERATION "REJECTION" MADE COMPLETELY INDEPENDENT.

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Some time ago Professor L. L. Dines* demonstrated the fact that while Sheffer's† five postulates for Boolean algebras in terms of "rejection" are independent in the ordinary sense that no one of the postulates is implied by the other four, they are not completely independent in the sense defined by Professor E. H. Moore‡ inasmuch as the negative of the first postulate implies the third, fourth, and fifth postulates of the set. It is the purpose of this paper to demonstrate that if the first postulate is replaced by one postulating a minimum of four instead of two distinct elements the resulting set of postulates is a completely independent set.§

Sheffer's five postulates concerning a system $\Sigma(K, |)$ are:

1. *There are at least two elements in K.*

2. *Whenever a and b are elements of K, $a \mid b$ is an element of K.*

Definition: $a' = a \mid a$.

3. *Whenever a and the indicated combinations of a are elements of K,* $(a')' = a$.

4. *Whenever a, b, and the indicated combinations of a and b are elements of K,*

$$a \mid (b \mid b') = a'.$$

* L. L. Dines. "Complete existential theory of Sheffer's postulates for Boolean algebras," this BULLETIN, vol. 21 (Jan., 1915), pp. 183–188.

† H. M. Sheffer, "A set of five postulates for Boolean algebras with applications to logical constants," *Transactions Amer. Math. Society*, vol. 14 (1913), pp. 481–488.

‡ E. H. Moore, "Introduction to a form of general analysis," New Haven Mathematical Colloquium, Yale University Press, page 82.

§ A set of m postulates is said to be ordinarily independent if no one of the m postulates is implied by the others. A set of m postulates is said to be completely independent if, and only if, there are no implicational relations existing among the properties defined either by the postulates as they stand or by the negatives of the postulates. For, if the truth or falsity of one postulate implies either that another postulate is true or that it is false, it would seem either that the two postulates are concerned with two aspects of the same fundamental property, or that there are two fundamental properties involved in such a manner that one of the postulates, at least, deals with both properties.

5. *Whenever a, b, c, and the indicated combinations of a, b and c are elements of K,*

$$[a \mid (b \mid c)]' = (b' \mid a) \mid (c' \mid a).$$

As stated above, the negative of the first postulate implies the third, fourth, and fifth postulates. For, if K contains less than two distinct elements, these three postulates are satisfied either evidently or vacuously according as the second postulate does or does not hold. This difficulty is no longer encountered, however, if, instead of assuming postulate 1, we assume the following.

1′. *There are at least four distinct elements in K.*

The truth of this statement is demonstrated by the exhibition of thirty-two systems having the $2^5 = 32$ possible characters $(+++++)$, $(++++-), \cdots$, $(-----)$, the ith sign being plus or minus according as the ith postulate is or is not satisfied.

There are two systems with K^{singular}, eight with K^{dual}, six with K^{triple}, and sixteen with $K^{\text{quadruple}}$. In each case the systems have been chosen with as few elements as possible.

In accordance with custom the result of combining elements is given by means of tables. For example, if K contains two elements l_1 and l_2 and if $l_1 \mid l_1 = l_1$, $l_1 \mid l_2 = l_1$, $l_2 \mid l_1 = l_2$, and $l_2 \mid l_2 = l_2$, this will be stated in the form

1	l_1	l_2
l_1	l_1	l_1
l_2	l_2	l_2

If $l_i \mid l_j$ does not give an element in K, this will be indicated by saying that $l_i \mid l_j = x$. The extension of this sort of table to systems containing more than two elements is obvious.

The thirty-two systems with their indicated characters follow:

K Singular.

System I_1. $(-++++)$ $l_1 \mid l_1 = l_1$.

System I_2. $(--+++)$ $l_1 \mid l_1 \neq l_1$.

K Dual.

System II_1. $(-+++-)$

1	l_1	l_2
l_1	l_1	l_1
l_2	l_2	l_2

System II_2. $(-++-+)$

1	l_1	l_2
l_1	l_1	l_2
l_2	l_2	l_2

System II_3.

$$(-+-++)\quad \begin{array}{c|cc} 1 & l_1 & l_2 \\ \hline l_1 & l_1 & l_1 \\ l_2 & l_1 & l_1 \end{array}$$

System II_4.

$$(-++--)\quad \begin{array}{c|cc} 1 & l_1 & l_2 \\ \hline l_1 & l_1 & l_2 \\ l_2 & l_1 & l_2 \end{array}$$

System II_5.

$$(-+--+)\quad \begin{array}{c|cc} 1 & l_1 & l_2 \\ \hline l_1 & l_1 & l_2 \\ l_2 & l_2 & l_1 \end{array}$$

System II_6.

$$(--+-+)\quad \begin{array}{c|cc} 1 & l_1 & l_2 \\ \hline l_1 & l_1 & l_2 \\ l_2 & x & l_2 \end{array}$$

System II_7.

$$(---++)\quad \begin{array}{c|cc} 1 & l_1 & l_2 \\ \hline l_1 & l_1 & x \\ l_2 & x & l_1 \end{array}$$

System II_8.

$$(----+)\quad \begin{array}{c|cc} 1 & l_1 & l_2 \\ \hline l_1 & l_2 & l_1 \\ l_2 & x & l_2 \end{array}$$

K Triple.

System III_1.

$$(-+-+-)\quad \begin{array}{c|ccc} 1 & l_1 & l_2 & l_3 \\ \hline l_1 & l_1 & l_2 & l_3 \\ l_2 & l_1 & l_1 & l_1 \\ l_3 & l_2 & l_1 & l_2 \end{array}$$

System III_2.

$$(--++-)\quad \begin{array}{c|ccc} 1 & l_1 & l_2 & l_3 \\ \hline l_1 & l_1 & l_2 & l_3 \\ l_2 & l_3 & l_3 & l_1 \\ l_3 & l_2 & x & x \end{array}$$

System III_3.

$$(-+---)\quad \begin{array}{c|ccc} 1 & l_1 & l_2 & l_3 \\ \hline l_1 & l_1 & l_2 & l_3 \\ l_2 & l_3 & l_1 & l_3 \\ l_3 & l_2 & l_3 & l_2 \end{array}$$

System III_4.

$$(--+--)\quad \begin{array}{c|ccc} 1 & l_1 & l_2 & l_3 \\ \hline l_1 & l_1 & l_2 & l_3 \\ l_2 & l_1 & l_2 & l_3 \\ l_3 & l_1 & x & l_3 \end{array}$$

System III_5.

$$(---+-)\quad \begin{array}{c|ccc} 1 & l_1 & l_2 & l_3 \\ \hline l_1 & l_1 & l_2 & x \\ l_2 & l_1 & l_1 & l_1 \\ l_3 & l_2 & l_1 & l_2 \end{array}$$

System III_6.

$$(-----)\quad \begin{array}{c|ccc} 1 & l_1 & l_2 & l_3 \\ \hline l_1 & l_1 & l_2 & x \\ l_2 & l_3 & l_1 & l_2 \\ l_3 & x & x & l_2 \end{array}$$

K Quadruple.

System IV_1.

$(++++++)$

1	l_1	l_2	l_3	l_4
l_1	l_2	l_1	l_4	l_3
l_2	l_1	l_1	l_1	l_1
l_3	l_4	l_1	l_4	l_1
l_4	l_3	l_1	l_1	l_3

System IV_2.

$(+++++-)$

1	l_1	l_2	l_3	l_4
l_1	l_1	l_1	l_1	l_1
l_2	l_2	l_2	l_2	l_2
l_3	l_3	l_3	l_3	l_3
l_4	l_4	l_4	l_4	l_4

System IV_3.

$(+++-+)$

1	l_1	l_2	l_3	l_4
l_1	l_1	l_1	l_1	l_1
l_2	l_1	l_2	l_1	l_1
l_3	l_1	l_1	l_3	l_1
l_4	l_1	l_1	l_1	l_4

System IV_4.

$(++-++)$

1	l_1	l_2	l_3	l_4
l_1	l_1	l_1	l_1	l_1
l_2	l_1	l_1	l_1	l_1
l_3	l_1	l_1	l_1	l_1
l_4	l_1	l_1	l_1	l_1

System IV_5.

$(+-+++)$

1	l_1	l_2	l_3	l_4
l_1	x	x	x	x
l_2	x	x	x	x
l_3	x	x	x	x
l_4	x	x	x	x

System IV_6.

$(+++--)$

1	l_1	l_2	l_3	l_4
l_1	l_1	l_2	l_3	l_4
l_2	l_1	l_2	l_3	l_4
l_3	l_1	l_2	l_3	l_4
l_4	l_1	l_2	l_3	l_4

System IV_7.

$(++-+-)$

1	l_1	l_2	l_3	l_4
l_1	l_2	l_2	l_1	l_1
l_2	l_1	l_1	l_1	l_1
l_3	l_1	l_1	l_1	l_1
l_4	l_1	l_1	l_1	l_1

System IV_8.

$(+-++-)$

1	l_1	l_2	l_3	l_4
l_1	l_1	l_1	x	x
l_2	l_2	l_2	x	x
l_3	x	x	x	x
l_4	x	x	x	x

System IV_9.

$(++--+)$

1	l_1	l_2	l_3	l_4
l_1	l_1	l_2	l_1	l_1
l_2	l_2	l_1	l_1	l_1
l_3	l_1	l_1	l_1	l_1
l_4	l_1	l_1	l_1	l_1

System IV_{10}.

$(+-+-+)$

1	l_1	l_2	l_3	l_4
l_1	l_1	l_2	x	x
l_2	x	l_2	x	x
l_3	x	x	x	x
l_4	x	x	x	x

System IV_{11}.

$(+--++)$

1	l_1	l_2	l_3	l_4
l_1	l_1	x	x	x
l_2	x	l_1	x	x
l_3	x	x	x	x
l_4	x	x	x	x

System IV_{12}.

$(++---)$

1	l_1	l_2	l_3	l_4
l_1	l_2	l_1	l_1	l_1
l_2	l_1	l_3	l_1	l_1
l_3	l_3	l_3	l_3	l_3
l_4	l_4	l_4	l_4	l_4

System IV_{13}.

$(+-+--)$

1	l_1	l_2	l_3	l_4
l_1	l_1	l_2	x	x
l_2	l_1	l_2	x	x
l_3	x	x	x	x
l_4	x	x	x	x

System IV_{14}.

$(+--+-)$

1	l_1	l_2	l_3	l_4
l_1	l_1	l_2	l_3	x
l_2	l_1	l_1	l_1	x
l_3	l_2	l_1	l_2	x
l_4	x	x	x	x

System IV_{15}.

$(+----+)$

1	l_1	l_2	l_3	l_4
l_1	l_2	l_1	x	x
l_2	x	l_2	x	x
l_3	x	x	x	x
l_4	x	x	x	x

System IV_{16}.

$(+-----)$

1	l_1	l_2	l_3	l_4
l_1	l_1	l_2	x	x
l_2	l_3	l_1	l_2	x
l_3	x	x	l_2	x
l_4	x	x	x	x

It may be of interest to note that a similar change in the first postulate of Bernstein's* set of four postulates in terms of the operator "rejection" also makes that set completely independent, as I have shown in an earlier paper,† and that furthermore this same change makes my own set of five postulates in terms of "exception"‡ completely independent (together with a change in the fifth postulate at a sacrifice of simplicity). It would be interesting to ascertain whether this postulation of a minimum of four distinct elements is sufficient to being about the complete independence of any

* B. A. Bernstein, "A set of four independent postulates for Boolean algebras," *Transactions Amer. Math. Society*, vol. 17 (1916), pp. 50–52.

† J. S. Taylor, "Complete existential theory of Bernstein's set of four postulates for Boolean algebras," *Annals of Mathematics*, second series, vol. 19, no. 1 (Sept., 1917), pp. 64–69.

‡ J. S. Taylor, "A set of five postulates for Boolean algebras in terms of the operation 'exception,'" *University of California Publications in Mathematics*, vol. 1 (April 12, 1920), pp. 241–248.

set of postulates for Boolean algebras where the remaining postulates of the set are already free from all implicational relationships among themselves.

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ROTATION SURFACES OF CONSTANT CURVATURE IN SPACE OF FOUR DIMENSIONS.

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1. IN space of four dimensions there are two special rotations each of which has circles for path curves. In this note I shall discuss the surfaces generated by these special rotations which have constant curvature. The first type is given by the equations

$$X = x \cos t - y \sin t, \quad Y = x \sin t + y \cos t, \qquad Z = z, \quad W = w. \tag{1}$$

This rotation leaves each point of the zw-plane invariant and any plane completely perpendicular to it is left invariant as a plane but not point for point. The rotation then is simply isomorphic with a rotation in the xy-plane.*

If the curve

$$x = x(s), \quad y = y(s), \quad z = z(s), \quad w = w(s), \tag{c}$$

where s denotes arc length measured from a fixed point, is rotated, equations (1) are the parametric equations of the surface generated. The parameter curves $s=$ const., $t=$ const. will be orthogonal if

$$xy' - x'y = 0 \quad \text{or} \quad y = kx, \tag{2}$$

where primes denote derivatives with respect to s. Hence *the meridian curves* (*orthogonal trajectories of the path curves*) *on a surface generated by rotation* (1), *lie in a 3-space which contains the absolutely invariant plane.* If the meridian curve

* Phillips and Moore, "Rotations in space of even dimensions," *Proceedings Amer. Academy*, vol. 55.