holm determinant is in the system of linear integral equations

$$(4) \qquad \begin{aligned} \psi_1 &= \varphi_1 + \int \kappa_{11}\varphi_1 + \int \kappa_{12}\varphi_2, \\ \psi_2 &= \varphi_2 + \int \kappa_{21}\varphi_1 + \int \kappa_{22}\varphi_2, \end{aligned}$$

in which the kernels κ_{12}, and κ_{22} are expressed in terms of the $3n$ functions $\alpha_i(x)$, $\beta_i(x)$, and $\gamma_i(x)$ as follows:

$$\kappa_{12} = \Sigma_1^n \alpha_i(x)\beta_i(y), \qquad \kappa_{22} = \Sigma_1^n \gamma_i(x)\beta_i(y).$$

If we substitute these values and multiply the second of the two equations by $\beta_j(x)$ and integrate with respect to x, we replace the above system by a system of the type (3) in which $\varphi_j = \int \beta_j(x)\varphi_2(x)dx$. The value of the function φ_2 can then be determined from the second of the equations (4). Since the Fredholm determinant of a system (3) is a bordered determinant, the Fredholm determinant of a system of the type of (4) would also be.

Obviously, these results can be extended by introducing the general range in place of the range I: $a \leqq x \leqq b$ and general classes of functions instead of continuous functions, and a general linear operator in place of integration in accordance with the postulates of Moore's general theory.

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ON THE COHERENCE OF CERTAIN SYSTEMS IN GENERAL ANALYSIS.

BY PROFESSOR A. D. PITCHER

IN a previous paper* attention was called to the property *coherence*, so named, of systems $(\mathfrak{Q}; \delta)$, where $\mathfrak{Q}$ is an abstract class of elements q and $\delta(q_1q_2)$ a distance function defined for every pair q_1q_2 of elements of $\mathfrak{Q}$. In the present paper it is shown that certain of the most notable of the systems of E. H. Moore's Introduction to General Analysis† which were devised without reference to coherence, or indeed without

* "On the foundations of the calcul fonctionnel of Fréchet," by A. D. Pitcher and E. W. Chittenden, *Transactions Amer. Math. Society*, vol. 19, No. 1, pp. 66–78.

† Cf. New Haven Mathematical Colloquium, Yale University Press, 1910. We refer to this memoir as I. G. A.

reference to any property of the range of the independent variable, must nevertheless be coherent.

Consider a system $(\mathfrak{P}; \Delta)$, i.e., an abstract class $\mathfrak{P}$ of elements p with a development Δ*, and the relation $K_{p_1 p_2 m}$† to which the development Δ gives rise. Two sequences $\{p_{1n}\}$ $\{p_{2n}\}$ are said to be connected in case for every integer m there is an integer n_m such that $n \geqq n_m$ implies $K_{p_{1n} p_{2n} m}$. An element p is said to be a limit of a sequence $\{p_n\}$, in notation $L_n p_n = p$, in case for every m there is an n_m such that $n \geqq n_m$ implies $K_{p_n p m}$.‡ A system $(\mathfrak{P}; \Delta)$, or a system $(\mathfrak{P}; K)$, is coherent if it is true that whenever two sequences $\{p_{1n}\}$ $\{p_{2n}\}$ are connected and p is a limit of the sequence $\{p_{1n}\}$ then p is also a limit of the sequence $\{p_{2n}\}$.

We wish to consider the systems $(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M})$§ where the class $\mathfrak{M}$ has the properties $D_1 \Delta K_2$ or the properties $L \Delta K_2$.‖

THEOREM. *The system* $(\mathfrak{P}; \Delta)$ *of a system* $(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M})$, *where the class* $\mathfrak{M}$ *has the properties* $D_1 \Delta K_2$ *or the properties* $L \Delta K_2$, *is a coherent system.*¶

Since $\{p_{1n}\}$ and $\{p_{2n}\}$ are connected there is a sequence $\{\mathfrak{P}^{\hat{m}_n \hat{l}_n}\}$ of the development Δ such that $L_n \hat{m}_n = \infty$ and such that for $n \geqq n_0$ the elements p_{1n} and p_{2n} each belong to the class $\mathfrak{P}^{\hat{m}_n \hat{l}_n}$. Since p is a limit of the sequence $\{p_{1n}\}$ there is a sequence $\{\mathfrak{P}^{\overline{m}_n \overline{l}_n}\}$, of the development Δ, such that $L_n \overline{m}_n = \infty$ and such that for n sufficiently large p and p_{1n} each belong to $\mathfrak{P}^{\overline{m}_n \overline{l}_n}$.

We must show that p is a limit of the sequence $\{p_{2n}\}$. Suppose this is not true. Then there is an integer m' such that for every n_0 there is an $n > n_0$ such that p and p_{2n} are not in the relation $K_{p p_{2n} m'}$. Thus by permitting n_0 to assume the values $1, 2, 3, \ldots, k, \ldots$ and choosing each time the smallest n available we secure a sequence $\{n_k\}$ such that for no k is the relation $K_{p p_{2n_k} m'}$ holding. There is then a sequence $\{p_{2n_k}\}$, a subsequence of $\{p_{2n}\}$, such that for no m exceeding

* Cf. I. G. A., § 75.

† I. G. A., § 77.

‡ Cf. "A contribution to the foundations of the calcul fonctionnel of Fréchet," by T. H. Hildebrandt, *Amer. Jour. of Mathematics*, vol. 34 (1912), pp. 237–290.

§ I. G. A., §§ 75–80.

‖ For properties D_1, L, Δ, K_2 of classes $\mathfrak{M}$ see I. G. A., §§ 15, 22, 79, 72.

¶ It should be noted that the property D implies the property D_1 and on that account the systems $(\mathfrak{A}; \mathfrak{P}; \Delta; \mathfrak{M})$, where $\mathfrak{M}$ has the properties $D \Delta K_2$, are included in the above theorem.

m' and for no k are p_{2n_k} and p in the same subclass $\mathfrak{P}^{ml}$ of the development Δ.

Now consider the element p and those functions of the developmental system $((\delta^{ml}))$ which are associated with the element p. That is, consider the system $((\delta^{mg}))$ where $(\mathfrak{P}^{mg})$ is the system of classes of stage m which contain p. Denote by θ^m the sum of the functions δ^{mg} which are associated with p at stage m. Then, by the first condition* on the developmental system,

$$L_m \theta_p^m = 1.$$

In case $\mathfrak{M}$ has either the property L or the property D_1 the function θ^m has the property $K_2\mathfrak{M}$†. Consider a function φ which has the property $K_2\mathfrak{M}$. There is a μ of $\mathfrak{M}$ such that for every positive number e there is an m_e such that $K_{p_1 p_2 m_e}$ implies $|\varphi_{p_1} - \varphi_{p_2}| \leqq e |\mu_{p_1}|$. Since $\{p_{1n}\}$ and $\{p_{2n}\}$ are connected it follows that for every e there is an n_e such that $n \geqq n_e$ implies $|\varphi_{p_{1n}} - \varphi_{p_{2n}}| \leqq e |\mu_{p_{1n}}|$. Since the sequence $\{p_{1n}\}$ has p for a limit and μ has the property $K_2\mathfrak{M}$, the sequence $\{\mu_{p_{1n}}\}$ of values is bounded. Thus for every e there is an n_e such that $|\varphi_{p_{1n}} - \varphi_{p_{2n}}| \leqq e$. Therefore $L_n \varphi_{p_{1n}} = L_n \varphi_{p_{2n}} = \varphi_p$.

If the conclusion concerning φ be applied to each function θ^m, we have

$$L_n \theta_{p_{1n}}^m = L_n \theta_{p_{2n}}^m = \theta_p^m.$$

Then for every m

$$L_k \theta_{p_{2n_k}}^m = \theta_p^m$$

and

$$L_m (L_k \theta_{p_{2n_k}}^m) = L_m \theta_p^m = 1.$$

According to the condition (1b) on the developmental system there is for every μ a function μ_0 such that for every e there is an m_e such that for $m \geqq m_e$ and for every p

$$\Sigma_h |\mu_{r_{mh}} \delta_p^{mh}| \leqq e |\mu_{0p}|.$$

Consider an element p_{2n_k}. Recall that $\theta^m = \Sigma_g \delta^{mg}$. For m sufficiently large no δ^{mg} is associated with a class $\mathfrak{P}^{ml}$ to which p_{2n_k} belongs (since δ^{mg} is associated with a class $\mathfrak{P}^{mg}$ to which p belongs). Since $\mathfrak{M}$ has the properties L, Δ, K_2 or the properties D_1, Δ, K_2 there is a function μ of $\mathfrak{M}$ which, for m sufficiently large and a number a properly chosen, is such

* I. G. A., § 78 (1a).
† I. G. A., § 72 (2).

that $a\mu_{r^{mg}} \geqq 1$, where r^{mg} is the representative element of any class $\mathfrak{P}^{mg}$ to which p belongs. Using such a function μ in the condition $(1b)$ on the developmental system and noting that the sequence $\{\mu_{0p_{2n_k}}\}$ is bounded, we see that for every e there is an m_e such that for $m \geqq m_e$ and for every k the value $\theta_{p_{2n_k}}{}^m$ does not exceed e. But this affords a contradiction to the conclusion reached above that

$$L_m(L_k\theta_{p_{2n_k}}{}^m) = 1.$$

Therefore the hypothesis that p is not a limit of the sequence $\{p_{2n}\}$ is contrary to fact.

These considerations may be extended to the infinite developments of Chittenden.*

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ON THE ADJOINT OF A CERTAIN MIXED EQUATION.

BY DR. R. F. BORDEN.

(Read before the American Mathematical Society December 30, 1919.)

CONSIDER a function of the form

(1) $F\{f(x)\} = \Delta f'(x) + a(x)f'(x) + b(x)\Delta f(x) + c(x)f(x),$

where $a(x)$, $b(x)$, and $c(x)$ are analytic functions of x, also $\Delta f(x) = f(x+1) - f(x)$, and $f'(x) = (d/dx)f(x)$. We will say that $G\{g(x)\}$ is the adjoint of $F\{f(x)\}$ if $G\{g(x)\} = 0$ is the condition that

$$\int \Sigma\, g(x)F\{f(x)\}dx = \Sigma M(x) + \int N(x)dx, \tag{2}$$

where Σ denotes an inverse of Δ.

This condition (2) is satisfied if

$$g(x)F\{f(x)\} = \frac{d}{dx}M(x)dx + \Delta N(x), \tag{3}$$

* Cf. "Infinite developments and the composition property $(K_{12}B_1)_*$ in general analysis," by E. W. Chittenden, *Rendiconti del Circolo Matem. di Palermo*, vol. 39 (1915), p. 21, §§ 19 and 21.