ON BORDERED FREDHOLM DETERMINANTS.

BY PROFESSOR T. H. HILDEBRANDT.

In a paper on projective transformations in function space, L. L. Dines* sets up certain expressions which he calls bordered Fredholm determinants, and minors. These determinants have properties which are analogous to the properties of Fredholm determinants. It is the purpose of the present note to point out that these bordered Fredholm determinants can be regarded as the Fredholm determinants of a system or a mixed system of linear integral equations, and that as a consequence their properties can be deduced from the corresponding properties of Fredholm determinants.

The bordered determinants arise in the question of the inversion of the transformation

$$\psi(x) = \frac{\alpha(x) + \beta(x)\varphi(x) + \int_a^b \gamma(x, y)\varphi(y)dy}{\delta + \int_a^b \epsilon(y)\varphi(y)dy}, \tag{1}$$

where δ is a constant, α, β, ϵ, φ and ψ are functions in the interval I: $a \leqq x \leqq b$, which we assume to be continuous, and $\gamma(x, y)$ is continuous in the square S: $a \leqq x \leqq b$, $a \leqq y \leqq b$. It is also assumed that β does not vanish in the interval I; as a consequence the question of the inversion of the transformation (1) is reducible to the inversion of the special transformation in which β is taken equal to unity on I. The existence of a unique inverse function φ for every ψ is equivalent to the existence of a unique solution of the system of equations

$$\begin{aligned} \psi_1(x) &= \varphi_1(x) + \alpha(x)\varphi_2 + \int \gamma(x, y)\varphi_1(y)dy,\dagger \\ \psi_2 &= \delta\varphi_2 + \int \epsilon(y)\varphi_1(y)dy, \end{aligned} \tag{2}$$

where φ_2 and ψ_2 are constants and $\varphi = \varphi_1/\varphi_2$ and $\psi = \psi_1/\psi_2$. The determination of φ_1 and φ_2 from this system is in turn

* Cf. *Transactions Amer. Math. Society*, vol. 20 (1919), pp. 45–65.
† Here, as in the sequel, we omit the limits of integration (a, b).

equivalent to the question of solving the system of linear integral equations

$$\psi_1 = \varphi_1 + \int \kappa_{11}\varphi_1 + \int \kappa_{12}\varphi_2,$$

$$\psi_2 = \varphi_2 + \int \kappa_{21}\varphi_1 + \int \kappa_{22}\varphi_2,$$

where

$$\kappa_{11}(x, y) = \gamma(x, y), \qquad \kappa_{12}(x, y) = \frac{\alpha(x)}{b - a}, \qquad \kappa_{21}(x, y) = \epsilon(y)$$

and

$$\kappa_{22}(x, y) = \frac{\delta - 1}{b - a}.$$

On account of the nature of the functions κ_{12}, κ_{21}, and κ_{22}, every determinant of the Fredholm expansion of the system $\begin{pmatrix} \kappa_{11}\kappa_{12} \\ \kappa_{21}\kappa_{22} \end{pmatrix}$ which contains more than one row or column involving κ_{12}, κ_{21}, κ_{22} vanishes identically, and the expansion reduces to

$$1 + \Sigma_n \frac{1}{n!}\int \cdots \int |\gamma(x_i, x_j)| dx_1 \cdots dx_n$$
$$+ \Sigma_n \frac{1}{n!}\int \cdots \int \begin{vmatrix} \delta - 1 & \alpha(x_i) \\ \epsilon(x_i) & \gamma(x_i, x_j) \end{vmatrix} dx_1 \cdots dx_n$$
$$(i, j = 1, \cdots, n),$$

which on combination becomes

$$1 + \Sigma_n \frac{1}{n!}\int \cdots \int \begin{vmatrix} \delta & \alpha(x_i) \\ \epsilon(x_j) & \gamma(x_i, x_i) \end{vmatrix} dx_1 \cdots dx_n \;\; (i, j = 1, \cdots, n),$$

which is exactly the expression for Dines's bordered determinant B.

B being then in fact a Fredholm determinant, it follows that we can derive the properties of B from those of Fredholm determinants. For instance, it is at once apparent that there is a product theorem between determinants B based on different functions. In particular since the product of the Fredholm determinants of the two systems $\begin{pmatrix} \kappa_{11}^{(1)}\kappa_{12}^{(1)} \\ \kappa_{21}^{(1)}\kappa_{22}^{(1)} \end{pmatrix}$ and $\begin{pmatrix} \kappa_{11}^{(2)}\kappa_{12}^{(2)} \\ \kappa_{21}^{(2)}\kappa_{22}^{(2)} \end{pmatrix}$ is the system

$$\begin{pmatrix} \kappa_{11}^{(1)} + \kappa_{11}^{(2)} + \int \kappa_{11}^{(1)}\kappa_{11}^{(2)} + \int \kappa_{12}^{(1)}\kappa_{21}^{(2)}, & \kappa_{12}^{(1)} + \kappa_{12}^{(2)} + \int \kappa_{11}^{(1)}\kappa_{12}^{(2)} + \int \kappa_{12}^{(1)}\kappa_{22}^{(2)} \\ \kappa_{21}^{(1)} + \kappa_{21}^{(2)} + \int \kappa_{21}^{(1)}\kappa_{11}^{(2)} + \int \kappa_{22}^{(1)}\kappa_{21}^{(2)}, & \kappa_{22}^{(1)} + \kappa_{22}^{(2)} + \int \kappa_{21}^{(1)}\kappa_{12}^{(2)} + \int \kappa_{22}^{(1)}\kappa_{22}^{(2)} \end{pmatrix}$$

we can substitute in this the corresponding values for the κ and find that the B-determinant of the functions δ_1, α_1, ϵ_1, γ_1 multiplied by the B-determinant of δ_2, α_2, ϵ_2, γ_2 is the B-determinant of $\delta_1\delta_2 + \int \epsilon_1\alpha_2, \alpha_2 + \int \gamma_1\alpha_2 + \alpha_1\delta_2, \epsilon_1 + \int \epsilon_1\gamma_2 + \delta_1\epsilon_2, \gamma_1 + \gamma_2 + \int \gamma_1\gamma_2 + \alpha_1\epsilon_2$, which is the result obtained by Dines by more cumbersome methods.

In order to complete the inversion problem it is necessary to introduce minors of the first order. There will be four of these in a system of two integral equations. If we set up these determinants for the special kernels under consideration and note as before that every determinant in the expansion which contains more than one row or column of the α, ϵ and δ is identically zero, we easily obtain the following expressions:

$$F_{11} = -\begin{vmatrix} \delta & \alpha(x) \\ \epsilon(y) & \gamma(x, y) \end{vmatrix} - \Sigma_n \frac{1}{n!}\int \cdots \int \begin{vmatrix} \delta & \alpha(x) & \alpha(x_i) \\ \epsilon(y) & \gamma(x, y) & \gamma(x_i, y) \\ \epsilon(x_j) & \gamma(x, x_j) & \gamma(x_i, x_j) \end{vmatrix} dx_1 \cdots dx_n \quad (i, j = 1, \cdots n),$$

$$F_{12} = -\frac{1}{b-a}\left[\alpha(x) + \Sigma_n \frac{1}{n!}\int \cdots \int \begin{vmatrix} \alpha(x) & \alpha(x_i) \\ \gamma(x, x_j) & \gamma(x_i, x_j) \end{vmatrix} dx_1 \cdots dx_n\right] \quad (i, j = 1, \cdots, n),$$

$$F_{21} = -\epsilon(y) - \Sigma_n \frac{1}{n!}\int \cdots \int \begin{vmatrix} \epsilon(y) & \gamma(x_i, y) \\ \epsilon(x_j) & \gamma(x_i, x_j) \end{vmatrix} dx_1 \cdots dx_n \quad (i, j = 1, \cdots, n),$$

$$F_{22} = -\frac{1}{b-a}\left[\delta - 1 + \Sigma_n \frac{1}{n!}\int \cdots \int \begin{vmatrix} \delta - 1 & \alpha(x_i) \\ \epsilon(x_j) & \gamma(x_i, x_j) \end{vmatrix} dx_1 \cdots dx_n\right] \quad (i, j = 1, \cdots, n).$$

By comparing with the results of Dines we see that

$$F_{11} = B_1, \qquad F_{12} = A/(b - a), \qquad F_{21} = E,$$
$$F_{22} = (D - B)/(b - a).$$

If now we write down the eight reciprocal relations governing the Fredholm determinant and its first minors for a system of equations, as for instance

$$F_0\kappa_{11} + F_{11} + \int F_{11}\kappa_{11} + \int F_{12}\kappa_{21} = 0$$

and seven others of a similar nature, and substitute therein the values for our special case, we get at once the reciprocal relations relative to B_1, A, E, B and D which Dines obtains. It is thus possible to make the discussion of the inversion of the projective transformation on the basis of the Fredholm determinants of a system of linear integral equations.

A more elegant approach to the same results is possible if we note that the system of equations (2) is a mixed system to which the results obtained by Moore* on general linear integral equations apply. Moore discusses not only the case when the functions which enter the equations have the same ranges but also when the ranges are different. Thus in the system under consideration, we really have two distinct ranges, the interval I: $a \leqq x \leqq b$, with the continuous functions φ_1, and ψ_1, and the operation of integration; and the range consisting of a single element with the functions φ_2, ψ_2, i.e., a single value, and the operation, the identity operation. By applying the methods sketched by Moore and reducing the results, we obtain at once as the Fredholm determinant of this system, the bordered determinant B, and as first minors, the expressions for B_1, A, E, and $D - B$. By substituting in the proper reciprocal relations, we arrive at the reciprocal relations obtained by Dines. This method has the advantage of not introducing any extraneous material, i.e., yielding at once the particular quantities which we desire.

This method is also useful in indicating a situation in which one obtains as Fredholm determinants a determinant bordered by n rows and columns. For if we replace the second range which consists of a single element, by a range which consists

* This BULLETIN, vol. 18 (1912), pp. 356 ff.

of a finite number m of elements, we obtain the equations

$$(3)\qquad \psi(x) = \varphi(x) + \int \kappa(x, y)\varphi(y)dy + \Sigma_1^m \kappa_h(x)\varphi_h,$$
$$\psi_g = \varphi_g + \int \lambda_g(y)\varphi(y)dy + \Sigma_1^m k_{gh}\varphi_h,$$

where φ and ψ are defined on $a \leqq x \leqq b$, and φ_g and ψ_g on $(1, \cdots, m)$. When we set up the Fredholm determinant of this system, we find that all the determinants involving more than m rows or columns of the kernels κ_h, λ_g and k_{gh} vanish identically, and by collecting the non-vanishing elements properly we obtain without much trouble as the Fredholm determinant of this system the expansion

$$|\delta_{gh} + k_{gh}| + \Sigma_n \frac{1}{n!}\int \cdots \int \begin{vmatrix} \delta_{gh}+k_{gh} & \kappa_g(x_i) \\ \lambda_h(x_j) & \kappa(x_i, x_j) \end{vmatrix} dx_1 \cdots dx_n \qquad \begin{pmatrix} g, h = 1, \cdots, m \\ i, j = 1, \cdots, n \end{pmatrix},$$

where $\delta_{gh} = 0$ if $g \neq h$ and unity if $g = h$. Obviously the first and higher ordered minors and the relation between these can be written out without much difficulty.

Systems of linear equations of this last type arise in a number of connections. For instance the mixed linear integral equat on

$$\psi(x) = \varphi(x) + \int \kappa(x, y)\varphi(y)dy + \Sigma_1^m \kappa_g(x)\varphi(x_g),$$

where x_h, $h=1, \cdots, m$, are m points on the interval $a \leqq x \leqq b$, can be reduced to such a system. We need only adjoin the m equations which we obtain by putting $x = x_h$ in this equation and set $\varphi(x_g) = \varphi_g$ in order to obtain such a system. It would follow then that the Fredholm determinant of this mixed integral equation can be written in the form

$$|\delta_{gh} + \kappa_g(x_h)| + \Sigma_n \frac{1}{n!}\int \cdots \int \begin{vmatrix} \delta_{gh}+\kappa_g(x_h) & \kappa_g(x_i) \\ \kappa(x_h, x_j) & \kappa(x_i, x_j) \end{vmatrix} dx_1 \cdots dx_n \qquad \begin{pmatrix} g, h = 1, \cdots, m \\ i, j = 1, \cdots, n \end{pmatrix}$$

with $\delta_{gh} = 0$ if $g \neq h$ and $\delta_{gg} = 1$.

Another situation in which we get such a bordered Fredholm determinant is in the system of linear integral equations

$$(4) \qquad \psi_1 = \varphi_1 + \int \kappa_{11}\varphi_1 + \int \kappa_{12}\varphi_2,$$
$$\psi_2 = \varphi_2 + \int \kappa_{21}\varphi_1 + \int \kappa_{22}\varphi_2,$$

in which the kernels κ_{12}, and κ_{22} are expressed in terms of the $3n$ functions $\alpha_i(x)$, $\beta_i(x)$, and $\gamma_i(x)$ as follows:

$$\kappa_{12} = \Sigma_1^n \alpha_i(x)\beta_i(y), \qquad \kappa_{22} = \Sigma_1^n \gamma_i(x)\beta_i(y).$$

If we substitute these values and multiply the second of the two equations by $\beta_j(x)$ and integrate with respect to x, we replace the above system by a system of the type (3) in which $\varphi_j = \int \beta_j(x)\varphi_2(x)dx$. The value of the function φ_2 can then be determined from the second of the equations (4). Since the Fredholm determinant of a system (3) is a bordered determinant, the Fredholm determinant of a system of the type of (4) would also be.

Obviously, these results can be extended by introducing the general range in place of the range I: $a \leqq x \leqq b$ and general classes of functions instead of continuous functions, and a general linear operator in place of integration in accordance with the postulates of Moore's general theory.

Ann Arbor, Mich.

ON THE COHERENCE OF CERTAIN SYSTEMS IN GENERAL ANALYSIS.

BY PROFESSOR A. D. PITCHER

In a previous paper* attention was called to the property *coherence*, so named, of systems ($\mathfrak{Q}$; δ), where $\mathfrak{Q}$ is an abstract class of elements q and $\delta(q_1q_2)$ a distance function defined for every pair q_1q_2 of elements of $\mathfrak{Q}$. In the present paper it is shown that certain of the most notable of the systems of E. H. Moore's Introduction to General Analysis† which were devised without reference to coherence, or indeed without

* "On the foundations of the calcul fonctionnel of Fréchet," by A. D. Pitcher and E. W. Chittenden, *Transactions Amer. Math. Society*, vol. 19, No. 1, pp. 66–78.

† Cf. New Haven Mathematical Colloquium, Yale University Press, 1910. We refer to this memoir as I. G. A.