Replace $2x$ by $x - y$ and $2y$ by $x + y$.

$$S(x - y) = S(x)C(y) - C(x)S(y).$$

If y is replaced by $- y$,

$$S(x + y) = S(x)C(y) + C(x)S(y).$$

Therefore

$$\begin{aligned} S^2(x + y) - S^2(x - y) &= 4S(x)C(x)S(y)C(y) \\ &= S(2x)S(2y). \end{aligned}$$

Replace $2x$ by $x + y$ and $2y$ by $x - y$. Then

$$S(x + y)S(x - y) = S^2(x) - S^2(y).$$

Substituting the relations found, it follows that

$$C(x + y)C(x - y) = C^2(x) + C^2(y) - 1,$$

that is, the odd component of $F(x)$ satisfies (5) while the even component (except for the factor $F(0)$) satisfies (4).

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THE EQUATION $ds^2 = dx^2 + dy^2 + dz^2$.

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1. THIS equation,* being of geometrical importance, has attracted several writers, including Serret (1847), Darboux (1873, 1887), de Montcheuil (1905), Salkowski (1909), Eisenhart (1911), and Pell (1918). The simple parametric solution of de Montcheuil, which is the starting point of considerable work in differential geometry, was not noticed by Serret or Darboux. It is somewhat remarkable that the latter overlooked this solution, as he himself makes use (Surfaces,

* Full references to earlier writers are given by Eisenhart, *Annals of Math.* (2), vol. 13 (1911), pp. 17–35. Pell's paper will be found ibid. (2), vol. 20, pp. 142–148. The substance of the present note, with the exception of section 8, is from an unpublished A.M. thesis, presented to the University of Washington in 1908, dealing with the general algebraic problems on which solutions of this kind depend. I wish to emphasize that § 8 was written only after I had read Pell's paper.

volume III, § 584, page 11) of the ' élégant artifice de calcul ' upon which the solution ultimately depends. Pell's recent solution follows from the same source. Here, without going into the theory of such equations of any degree which may be similarly solved, we shall merely show that the solutions of de Montcheuil and Pell are contained in another, from which any number of solutions of the same kind, viz., free from quadratures, may be found by inspection. Some of these may possibly be of use in geometry as are the known solutions.

We remark, however, that this equation is the simplest of a very wide class, to which the device given below is applicable. For the gist of that device lies in finding a rational integral algebraic function of several variables which reproduces itself in form with respect to multiplication, and the simplest such function, as remarked in 1202 by Fibonacci, is a sum of two squares. From one point of view the next simplest cases are Euler's four-square, and Degen's eight-square theorems. In connection with differential equations, Euler's theorem is of particular interest at the present time, giving, as will be shown elsewhere, a simple parametric solution of the fundamental equation in the Einstein-Grossmann theory of generalized relativity and gravitation.

And to call attention to an algebraic problem of importance, we may mention that the next and much more difficult class of differential equations solvable by algebraic methods analogous to that illustrated here, depends upon those algebraic forms which are transformable into a power of themselves by an algebraic substitution on the variables. The finding of all such forms, as was strongly emphasized by Eisenstein, is of the first importance arithmetically. Beyond a few isolated functions, such as the discriminant of a binary cubic (Eisenstein), and the Hessian of this discriminant (Cayley), little progress has been made toward a complete solution. The problem on several occasions attracted Cayley. It seems singular that arithmeticians have ignored Eisenstein's lead, especially after the beautiful uses which he made of his own result in his investigations on the binary cubic representations of integers. The principal object of this note is to attract attention once more to this problem, in showing that even its simplest solutions are also of use elsewhere. We may add that the algebraic problem seems to present great difficulties.

2. Fibonacci's result gives the identity

$$(1)\quad (\alpha\varphi - \beta\psi)^2 + (\alpha\psi + \beta\varphi)^2 = (\alpha\varphi + \beta\psi)^2 + (\alpha\psi - \beta\varphi)^2.$$

Hence a solution of

$$(2)\qquad ds^2 = dx^2 + dy^2 + dz^2$$

is given by, ($i = \sqrt{-1}$),

$$(3)\qquad \begin{aligned} s &= \int(\alpha\varphi - \beta\psi)dt, & x &= \int(\alpha\varphi + \beta\psi)dt, \\ y &= i\int(\alpha\psi + \beta\varphi)dt, & z &= \int(\alpha\psi - \beta\varphi)dt, \end{aligned}$$

in which α, β, φ, ψ now denote functions of a parameter t, arbitrary except as to obvious restrictions of analyticity in some domain.

3. For most purposes it is desirable to have (3) free from quadratures. Let f denote an arbitrary function of t, and f', f'', $\cdots$, $f^{(n)}$ its successive t-derivatives. Then it is clear that, on repeated integrations by parts,

$$\int f^{(n)} t^{n-c} dt$$

wherein $c \not> n$ is an integer > 0, is readily reducible to a form free from all integral signs. Thus

$$\int f''' t^2 dt = t^2 f'' - 2tf' + 2f, \qquad \int f''' t dt = tf'' - f',$$

etc., and it is unnecessary to write out the general formula. Hence if g is a polynomial of degree $n - c$ in t, $\int f^{(n)} g dt$ is at once reducible to a form free from quadratures.

Applying this remark to (3), we choose for α, β polynomials in t of respective degrees a, b, and replace φ, ψ by $\varphi^{(c+m)}$, $\varphi^{(c+n)}$, where c denotes the greater of a, b, or if $a = b$, either; a, b, m, n are integers > 0, and φ, ψ arbitrary functions of t, analytic in some domain. Upon performing the integrations by parts as indicated, (3) is reduced to a form free from quadratures. If in the result only derivatives of either function φ, ψ appear, an obvious change of notation will reduce the solution to one containing φ, ψ and their successive derivatives. Or this reduction may be obviated in the first place by assigning either a, b or m, n in advance, and noting by inspection the least values for the unassigned pair which will give a result of the desired kind.

4. A slightly different way of obtaining solutions free from quadratures depends upon the following obvious remark.

Write

$$\alpha = \sum_{r=1}^{\rho} A_r f^{(a_r)}, \qquad \beta = \sum_{s=1}^{\sigma} B_s g^{(b_s)},$$

and $\varphi = P$, $\psi = Q$, where A_r, B_s, P, Q are polynomials in t of respective degrees m_r, n_s, p, q, and f, g are arbitrary functions of t. Then each of $\alpha\varphi \pm \beta\psi$, $\alpha\psi \pm \beta\varphi$ in (3) is of the form

$$\sum_{r=1}^{\rho} \overline{A}_r f^{(a_r)} + \sum_{s=1}^{\sigma} \overline{B}_s g^{(b_s)}, \tag{4}$$

wherein $\overline{A}_r$, $\overline{B}_s$ are polynomials in t, some of which may reduce to constants, whose degrees are linear functions of the m_r, n_s, p, q. Obviously for either the a_r, b_s or the m_r, n_s, p, q preassigned, the other set of values may be assigned by inspection (in any infinity of ways) so that, as in § 3, the values of α, β, φ, ψ above given furnish an infinity of solutions of (2) free from quadratures.

5. To illustrate § 3, we choose $\alpha = 1$, $\beta = t$ in (3), at the same time replacing φ, ψ by φ'', ψ''. On integrating the resulting forms of (3) by parts as indicated, we find

$$s = \varphi' - t\psi' + \psi, \qquad x = \varphi' + t\psi' - \psi,$$
$$iy = \varphi - t\varphi' - \psi', \qquad z = \varphi - t\varphi' + \psi',$$

which is de Montcheuil's solution. Discussions of the generality of this solution, with applications to geometry, will be found in the papers of de Montcheuil, Salkowski and Eisenhart.

6. In illustration of § 4 it is seen at a glance that

$$\begin{aligned} \alpha &= (\alpha_1 t + \beta_1)f''' + \gamma_1 g'', & \varphi &= p_1 t + q_1, \\ \beta &= (\alpha_2 t + \beta_2)f''' + \gamma_2 g'', & \psi &= p_2 t + q_2, \end{aligned} \tag{5}$$

where the α_i, β_i, γ_i, p_i, q_i $(i = 1, 2)$ are independent of t, give a solution (3) free from quadratures. On performing the integrations by parts we get such a solution involving the ten arbitrary constants $\alpha_1, \cdots, q_2$; and on assigning special values to these constants an infinity of solutions. One of the latter is Pell's. Instead of directly assigning the values of $\alpha_1, \cdots, q_2$ which give Pell's solution, we shall briefly examine the more general solution, showing by other means that it actually contains Pell's.

7. Substituting the values (5) in (3), reducing the $\alpha\varphi \pm \beta\psi$, $\alpha\psi \pm \beta\varphi$ to the form (4), and integrating by parts, we get

$$\begin{aligned} s = (A_1t^2 + B_1t + C_1)f'' - (2A_1t + B_1)f' \\ + 2A_1f + (K_1t + L_1)g' - K_1g, \end{aligned} \tag{6}$$

and precisely similar forms for x, $-iy$, z with the respective suffixes 2, 3, 4 in place of 1, where the twenty constants $A_1, \cdots, L_4$ are as follows:

$$A_1 = \alpha_1p_1 - \alpha_2p_2, \qquad B_1 = \alpha_1q_1 - \alpha_2q_2 + \beta_1p_1 - \beta_2p_2,$$
$$C_1 = \beta_1q_1 - \beta_2q_2,$$
$$A_4 = \alpha_1p_2 - \alpha_2p_1, \qquad B_4 = \alpha_1q_2 - \alpha_2q_1 + \beta_1p_2 - \beta_2p_1,$$
$$C_4 = \beta_1q_2 - \beta_2q_1,$$

and the rest are obtained from these thus: the letters with suffix 2 from those with suffix 1 by changing all the signs to $+$, similarly for those with suffix 3 from 4; K_i, L_i from A_i, C_i respectively on replacing α by γ, β by γ respectively ($i = 1, 2, 3, 4$). From these twenty constants we find by a straightforward calculation that a necessary and sufficient condition that four functions of the form (6) shall give a solution of (2) is that the following indeterminate system of twelve equations be satisfied:

$$\Lambda_1^2 + \Lambda_3^2 = \Lambda_2^2 + \Lambda_4^2,$$

in which Λ represents A, C, K or L;

$$\Lambda_1Z_1 + \Lambda_3Z_3 = \Lambda_2Z_2 + \Lambda_4Z_4,$$

where $(\Lambda, Z) = (A, B), (B, C), (C, L), (K, L)$, or (A, K);

$$B_1^2+B_3^2+2(A_1C_1+A_3C_3) = B_2^2+B_4^2+2(A_2C_2+A_4C_4),$$
$$A_1L_1+A_3L_3+B_1K_1+B_3K_3 = A_2L_2+A_4L_4+B_2K_2+B_4K_4,$$
$$B_1L_1+B_3L_3+C_1K_1+C_3K_3 = B_2K_2+B_4L_4+C_2K_2+C_4L_4.$$

The similarity of these conditions to those occurring in the Gaussian theory of the composition of binary quadratic forms is noticeable, and may be traced to the common source that the norm of an algebraic integer, here quadratic, is self-reproductive in form with respect to multiplication. By various artifices the set of twelve can be reduced to more elegant forms; but as the interest of these is chiefly arithmetical, we pass them over.

8. It is easily seen that the indeterminate set of § 7 is satisfied by

$$A_1 = B_1 = C_1 = 0; \quad K_1 = 0, \quad L_1 = 1;$$

$$A_2 = -\tfrac{1}{2}, \quad B_2 = 0, \quad C_2 = \tfrac{1}{2}; \quad K_2 = -1; \quad L_2 = 0;$$

$$A_3 = \tfrac{1}{2}, \quad B_3 = 0, \quad C_3 = \tfrac{1}{2}; \quad K_3 = 1, \quad L_3 = 0;$$

$$A_4 = 0, \quad B_4 = 1, \quad C_4 = 0; \quad K_4 = 0, \quad L_4 = 1.$$

Putting these in the set (6), we get a particular solution of (2),

$$s = g', \qquad z = tf'' - f' + g',$$

$$x = \frac{1 - t^2}{2} f'' + tf' - f - tg' + g,$$

$$-iy = \frac{1 + t^2}{2} f'' - tf' + f + tg' - g,$$

which is Pell's solution.

9. The extension of solutions of this kind to those containing any number of constants connected by sets of identities is immediate and need not be followed farther here. But we may briefly consider why it is possible, from the present point of view, to find solutions of (2) free from quadratures. A little consideration will show the ultimate source to lie in the fact that in (3) the integrands are bilinear in (α, β), (φ, ψ). This in turn is referable to the fact that Fibonacci's identity is the simplest case of Lagrange's theorem, viz., the norm of any algebraic integer reproduces itself with respect to multiplication. In Fibonacci's identity the integer is quadratic; taking the simplest case in a cubic field, we find, in exactly the same way, solutions free from quadratures for

$$dx^3 + dy^3 + dz^3 - 3dxdydz = dX^3 + dY^3 + dZ^3 - 3dXdYdZ,$$

the differential form on the left being the norm of $dx + \omega dy + \omega^2 dz$, where ω is a complex cube root of unity. Until such equations present themselves in geometry or elsewhere there is little use in writing out their solutions. But we may glance at a trigonometric device which frequently is applicable when the integrands are quadratic. It is of interest in the simplest case because it leads to a famous solution due to Euler, also to the solution of a special case (constant coefficients), of an important equation considered by Weingarten and others. It is interesting to note that these solutions correspond to Lagrange's theorem for the general quadratic integer, so that in a sense they are generalizations of (2), which depends upon a special quadratic integer, viz., the complex unit i.

10. For a, b, k constants, φ, ψ arbitrary functions of t, we have the identities

$$\int (\varphi''' + k^2\varphi') \sin kt dt = \varphi'' \sin kt - k\varphi' \cos kt,$$

$$\int (\varphi''' + k^2\varphi') \cos kt dt = \varphi'' \cos kt + k\varphi' \sin kt,$$

$$(\varphi^2 + a\varphi\psi + b\psi^2)^2 = \Phi^2 + a\Phi\Psi + b\Psi^2,$$

where $\Phi = \varphi^2 - b\psi^2$, $\Psi = 2\varphi\psi + a\psi^2$. Hence, on putting $\alpha \equiv f''' + 4f'$, where f is an arbitrary function of t, and

$$\varphi = \sqrt{\alpha} \sin t, \qquad \psi = \sqrt{\alpha} \cos t,$$

we find, on integrating by means of the first identities, that the solution of

$$ds^2 = du^2 + adudv + bdv^2$$

given by

$$s = \int (\varphi^2 + a\varphi\psi + b\psi^2)dt,$$

$$u = \int (\varphi^2 - b\psi^2)dt, \qquad v = \int (2\varphi\psi + a\psi^2)dt,$$

is

$$2s = 4(b + 1)f + [2(b - 1) \sin 2t - 2a \cos 2t]f' + [(b + 1) + a \sin 2t + (b - 1) \cos 2t]f'',$$

$$2u = 4(1 - b)f - 2(1 + b)f' \sin 2t + [(1 - b) - (1 + b) \cos 2t]f'',$$

$$2v = 4af + (2a \sin 2t - 4 \cos 2t)f' + (a + 2 \sin 2t + a \cos 2t)f''.$$

On putting $a = 0$, $b = 1$ we get for a solution of Euler's equation

$$ds^2 = du^2 + dv^2$$

a form equivalent to his.

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