Then we may take n so great that $\xi_i^{(n)}$ contains every function in Σ_i, and then, since $\lambda_i^{(n)}$ is the least norm, we must have $\lambda_i^{(n)} < \epsilon$, and consequently $\lambda_i < \epsilon$. Hence

THEOREM IV. *A necessary and sufficient condition that a normalized system* $[\varphi]$ *be essentially linearly dependent is that* $\lambda_i = 0$ *for some* i.

Theorems II and IV give

THEOREM V. *A necessary and sufficient condition that a system* $[\varphi]$ *have an adjoint is that it be essentially linearly independent.*

THE UNIVERSITY OF OREGON,
November, 1919.

ON CERTAIN RELATED FUNCTIONAL EQUATIONS.

BY DR. W. HAROLD WILSON.

(Read before the American Mathematical Society December 27, 1917.)

§ 1. *Introduction.*

THIS paper treats of the relationships which exist between certain functional equations. In § 2, the equations

$$(1) \qquad S(x-y) = S(x)C(y) - C(x)S(y),$$

and

$$(2) \qquad C(x-y) = C(x)C(y) - k^2S(x)S(y)$$

are considered individually and as a system. It is shown that (1) and (2) have their solutions in common if $C(x)$ is an even function and $S(x) \not\equiv 0$. As a consequence, it is shown that if $k \neq 0$, then

$$S(x) = [F(x) - F(-x)]/2k, \text{ and } C(x) = [F(x) + F(-x)]/2,$$

where $F(x+y) = F(x)F(y)$. If $k = 0$ and $S(x) \not\equiv 0$, $C(x) \equiv 1$ and

$$S(x+y) = S(x) + S(y).$$

The work at this point is very closely allied to that of

Jensen* on the system of equations

$$S(x+y) = S(x)C(y) + C(x)S(y),$$
$$C(x+y) = C(x)C(y) + c^2S(x)S(y).$$

This system with $c^2 = -1$ has been discussed† by Tannery, Osgood, and Van Vleck and H'Doubler.

The equation satisfied by $F(x)$ has been discussed‡ by Cauchy, Vallée Poussin, Van Vleck and H'Doubler, and others. The solution analytic over the complex x-plane has been shown to be $F(x) = e^{cx}$, where c is an arbitrary constant. Vallée Poussin (loc. cit.) has shown that if $F(x)$ is bounded in an interval $(0, \epsilon)$ along the real axis, the solution when x is real is e^{cx}, where c is an arbitrary constant. It follows, therefore, that if $F(x)$ is bounded in an interval $(0, \epsilon)$ along the real axis and in an interval $(0, \epsilon')$ along the axis of imaginaries, and if $x = u + v\sqrt{-1}$ where u and v are real, then

$$F(x) = F(u + v\sqrt{-1}) = F(u)F(v\sqrt{-1}) = e^{cu+dv},$$

where c and d are arbitrary constants, since $\varphi(v) = F(v\sqrt{-1})$ also satisfies the given equation. The solution continuous along any line in the complex x-plane also takes the last named form.

The equation $S(x+y) = S(x) + S(y)$ has been discussed by many writers of whom we may mention§ Cauchy, Darboux and Vallée Poussin. The solution $S(x)$ analytic over the complex x-plane is $S(x) = cx$, where c is an arbitrary constant. The solution bounded in an interval $(0, \epsilon)$ on the real axis and in an interval $(0, \epsilon')$ along the axis of imaginaries is, if $x = u + v\sqrt{-1}$ as above, $S(x) = cu + dv$, where c and d are arbitrary constants. The solution continuous along a line in the complex x-plane also takes this last form.

Andrade|| has found the continuous solution of

$$(3) \qquad C(x+y) + C(x-y) = 2C(x)C(y),$$

* *Tidsskrift for Mathematik*, vol. 2, ser. 4 (1878), p. 149.

† Tannery, Fonctions d'une Variable, 1886, p. 147. Osgood, Lehrbuch der Funktionentheorie, 1912, p. 582. Van Vleck and H'Doubler, *Transactions Amer. Math. Society*, vol. 17 (1916), p. 30.

‡ Cauchy, Cours d'Analyse (1821), Chapter 5. Vallée Poussin, Cours d'Analyse infinitésimale (1903), p. 30. Van Vleck and H'Doubler, loc. cit., p. 20.

§ Cauchy, loc. cit. Darboux, *Math. Annalen*, vol. 17 (1880), p. 56. Vallée Poussin, loc. cit.

|| *Bulletin de la Société Math. de France*, vol. 28 (1900), p. 58.

when x and y are real, by the use of integrals. Carmichael* has shown that equations (3) and

$$C(x+y)C(x-y) = C^2(x) + C^2(y) - 1, \qquad C(0) = 1, \tag{4}$$

to which all equations of the form

$$\bar{C}(x+y)\bar{C}(x-y) = \bar{C}^2(x) + \bar{C}^2(y) - m^2, \qquad m \neq 0,$$

can be reduced by the substitution $\bar{C}(x) = \bar{C}(0)C(x)$, have their uniform analytic solutions in common if $C(0) \neq 0$ in (3). In § 3 it is proved that all solutions of (3) and (4), except the trivial solution $C(x) \equiv 0$ of (3), are common. It is then proved that (3) and (4) have their solutions in common with the solutions of the system (1) and (2), where $S(x)$ is introduced in the discussion of (3) and (4) by the definition

$$S(x) = \tfrac{1}{2}[C(x-b) - C(x+b)]$$

and is chosen to be not identically zero unless $C(x) \equiv 1$.

Carmichael (loc. cit.) also discussed the uniform analytic solutions of

$$S(x+y)S(x-y) = S^2(x) - S^2(y). \tag{5}$$

The periodic solutions of equations (4) and (5) were discussed by Van Vleck and H'Doubler (loc. cit., page 30), with the added relation $C^2(x) + S^2(x) = 1$. In § 4 no relation is assumed between the functions of (4) and (5) but it is proved that $S(x)$ and $C(x)$ defined by the relation $C(x) = \frac{1}{2}[S(x+a) - S(x-a)]$, where $S(a) \neq 0$, satisfy equations (1) and (2).

In §§ 5, 6, 7, the equations

$$g(x+y) - g(x-y) = 2k^2S(x)S(y), \qquad k \neq 0, \tag{6}$$

$$f(x+y) + f(x-y) = 2f(x)C(y), \tag{7}$$

and

$$S(x+y) - S(x-y) = 2C(x)S(y) \tag{8}$$

are discussed and their relationship with the preceding equations is exhibited.

Finally, in § 8, the general equation

$$\alpha F(x+y)F(x-y) = \beta\varphi(\mu x) + \gamma\psi(\eta y) + \delta \qquad (\alpha, \beta, \gamma \neq 0), \tag{9}$$

* *American Mathematical Monthly*, vol. 16 (1909), p. 180.

is discussed and its relation to the preceding equations is exhibited. This is analogous to Pexider's* generalization of the Cauchy equations.

It should be noted that the theorems of this paper are independent of the restrictions which may be imposed to obtain particular solutions of the equations.

§ 2. *Equations* (1) *and* (2).

THEOREM I. *If* $S(x)$ *and* $C(x)$ *satisfy equation* (1) *and* $S(x) \not\equiv 0$, *then the odd component of* $C(x)$ *is a constant multiple of* $S(x)$, *and* $S(x)$ *and the even component* $E(x)$ *of* $C(x)$ *satisfy equations* (1) *and* (2) *simultaneously, where*

$$k^2 = \frac{E^2(a) - E(0)}{S^2(a)} \qquad \textit{and} \qquad S(a) \neq 0.$$

Interchange x and y. Then

$$S(y - x) = -S(x - y),$$

that is, $S(x)$ is an odd function and $S(0) = 0$. Discarding the trivial solution $S(x) \equiv 0$, there is some value a of x such that $S(a) \neq 0$.

Let $C(x) = E(x) + O(x)$, where $E(x)$ is even and $O(x)$ is odd. Equation (1) becomes

$$(1') \quad S(x - y) = S(x)E(y) + S(x)O(y) - E(x)S(y) - O(x)S(y).$$

Replace x by $-x$ and y by $-y$ in (1′) and add the equation thus obtained to equation (1′). Then

$$S(x)O(y) - O(x)S(y) = 0.$$

If $y = a$,

$$O(x) \equiv \frac{O(a)}{S(a)} S(x).$$

Equation (1′) now becomes

$$S(x - y) = S(x)E(y) - E(x)S(y),$$

whence

$$E(x)S(y) = S(x)E(y) - S(x - y).$$

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* *Monatshefte für Mathematik und Physik*, vol. 14 (1903), p. 293.

Now replace x by $x - y$ and y by a.

$$
\begin{aligned}
E(x-y)S(a) &= S(x-y)E(a) - S(x-y-a) \\
&= [E(a)S(x) - S(x-a)]E(y) \\
&\qquad - [E(a)E(x) - E(x-a)]S(y) \\
&= S(a)E(x)E(y) - [E(a)E(x) - E(x-a)]S(y).
\end{aligned}
$$

Interchange x and y and compare the equation so obtained with the last equation. It is obvious that

$$[E(a)E(x) - E(x-a)]S(y) = [E(a)E(y) - E(y-a)]S(x).$$

Let $y = a$. Then

$$E(a)E(x) - E(x-a) = k^2 S(a)S(x),$$

where

$$k^2 = \frac{E^2(a) - E(0)}{S^2(a)} .$$

Therefore

$$E(x-y) = E(x)E(y) - k^2 S(x)S(y).$$

THEOREM II. *If $S(x)$ and $C(x)$ satisfy equation* (2) *and if $C(x) \not\equiv$ constant, then $k \neq 0$, $S(x) \not\equiv 0$, and $S(x)$ and $C(x)$ satisfy equations* (1) *and* (2) *simultaneously.*

Interchange x and y. Then

$$C(y-x) = C(x-y),$$

that is, $C(x)$ is even. If $S(x) \equiv 0$ or if $k = 0$, then

$$C(x+y) = C(x)C(-y) = C(x)C(y) = C(x-y)$$

and if $y = x$,

$$C(2x) \equiv C(0),$$

that is, $C(x)$ is identically constant (0 or 1). Setting aside this trivial case, $k \neq 0$, there is some value a of x such that $S(a) \neq 0$, and there is some value b of x such that $C(b) \neq 0$.

Set $S(x) = E_1(x) + O_1(x)$, where $E_1(x)$ is even and $O_1(x)$ is odd. Equation (2) becomes

$$
\begin{aligned}
C(x-y) = C(x)C(y) - k^2 [&E_1(x)E_1(y) \\
&+ E_1(x)O_1(y) + E_1(y)O_1(x) + O_1(x)O_1(y)].
\end{aligned}
$$

Replace x by $-x$ and y by $-y$ and subtract the equation so obtained from the last equation. Then

$$E_1(x)O_1(y) + E_1(y)O_1(x) = 0.$$

If $O_1(x) \equiv 0$, then, as above, $C(x) \equiv$ constant. Discarding this trivial case, there is some value $\bar{a}$ of x such that $O_1(\bar{a}) \neq 0$. Let $x = y = \bar{a}$. Then

$$2E_1(\bar{a})O_1(\bar{a}) = 0$$

whence $$E_1(\bar{a}) = 0.$$

Now let $y = \bar{a}$ and let x vary;

$$O_1(\bar{a})E_1(x) = 0$$

whence $$E_1(x) \equiv 0$$

and $S(x)$ is odd and $S(0) = 0$. In (2) let $y = 0$ and $x = b$. Then $C(0) = 1$ and hence if $y = x$ in (2),

$$C^2(x) - k^2S^2(x) = 1.$$

Replace y by $-y$ in equation (2). Then

$$C(x + y) = C(x)C(y) + k^2S(x)S(y).$$

Now replace x by $x + a$ and y by a in equation (2);

$$\begin{aligned} k^2S(a)S(x + a) &= C(a)C(x + a) - C(x) \\ &= C^2(a)C(x) + k^2C(a)S(a)S(x) - C(x) \\ &= [C^2(a) - 1]C(x) + k^2C(a)S(a)S(x) \\ &= k^2S^2(a)C(x) + k^2C(a)S(a)S(x) \end{aligned}$$

whence $S(x + a) = S(a)C(x) + C(a)S(x)$.
Finally, replace x by $x - y$ and y by a in equation (2). It follows that

$$\begin{aligned} k^2S(a)S(x - y) &= C(a)C(x - y) - C(x - y - a) \\ &= C(a)C(x)C(y) - k^2C(a)S(x)S(y) \\ &\qquad - C(x)C(y + a) + k^2S(x)S(y + a) \\ &= k^2S(x)[S(y + a) - C(a)S(y)] \\ &\qquad - C(x)[C(y + a) - C(a)C(y)] \\ &= k^2S(a)S(x)C(y) - k^2S(a)C(x)S(y). \end{aligned}$$

Therefore

$$S(x - y) = S(x)C(y) - C(x)S(y).$$

THEOREM III. *Suppose that* $S(x)$ *and* $C(x)$ *satisfy equations* (1) *and* (2) *simultaneously. If* $k \neq 0$,

$$S(x) = \frac{F(x) - F(-x)}{2k} \qquad \textit{and} \qquad C(x) = \frac{F(x) + F(-x)}{2},$$

where $F(x + y) = F(x)F(y)$. *If* $k = 0$, *and* $S(x) \not\equiv 0$,

$$S(x + y) = S(x) + S(y) \qquad \textit{and} \qquad C(x) \equiv 1.$$

By interchanging x and y, we see that $S(x)$ is odd and $C(x)$ is even. Replace y by $-y$. Then

$$S(x + y) = S(x)C(y) + C(x)S(y),$$

$$C(x + y) = C(x)C(y) + k^2S(x)S(y).$$

The function $F(x) = C(x) + kS(x)$ satisfies the equation $F(x + y) = F(x)F(y)$. Now $F(-x) = C(x) - kS(x)$ and therefore

$$C(x) = \tfrac{1}{2}[F(x) + F(-x)] \quad \text{and} \quad S(x) = \frac{1}{2k}[F(x) - F(-x)]$$

if $k \neq 0$.

If $k = 0$, we have seen from equation (2) that $C(x) \equiv C(0)$. If $S(x) \not\equiv 0$, then by equation (1) $C(x) \not\equiv 0$. But if $y = 0$ in equation (2), $C(x) = C(x)C(0)$ whence if $C(b) \neq 0$, $C(0) = 1$ and (1) becomes $S(x - y) = S(x) - S(y)$. Replace y by $-y$. Then $S(x + y) = S(x) + S(y)$.

§ 3. *Equations* (3) *and* (4).

THEOREM I. *If* $C(x)$ *satisfies equation* (3) *and is not identically zero, then it satisfies equation* (4), *and, conversely, if* $C(x)$ *satisfies equation* (4), *then it satisfies equation* (3).

Since, by hypothesis, $C(x) \not\equiv 0$, there is some value b of x such that $C(b) \neq 0$. If in (3) $x = b$, $y = 0$, then $C(0) = 1$. If in (3) $y = x$,

$$C(2x) = 2C^2(x) - 1.$$

Replace x by $x + y$ and y by $x - y$. Then

$$\begin{aligned} 2C(x + y)C(x - y) &= C(2x) + C(2y) \\ &= 2C^2(x) + 2C^2(y) - 2. \end{aligned}$$

Therefore $C(x)$ satisfies equation (4) if it satisfies equation (3) and does not vanish identically.

In (4) let $y = x$. Then

$$C(2x) = 2C^2(x) - 1.$$

Multiply equation (4) by 2 and apply the last equation. Then

$$2C(x + y)C(x - y) = C(2x) + C(2y).$$

Replace $2x$ by $x + y$ and $2y$ by $x - y$. It follows that

$$C(x + y) + C(x - y) = 2C(x)C(y),$$

that is, $C(x)$ satisfies (3) if it satisfies (4).

THEOREM II. *If $C(x)$ satisfies equation* (3) *or equation* (4), *then $C(x)$ and $S(x) = \frac{1}{2}[C(x - b) - C(x + b)]$, where b is so chosen, when possible, that $C(2b) \neq 1$, satisfy equations* (1) *and* (2) *simultaneously, where, if*

$$S(x) \not\equiv 0, \qquad k^2 = \frac{2}{C(2b) - 1}.$$

If $C(x) \equiv 0$ or $C(x) \equiv 1$ the theorem is obvious. Discarding these trivial solutions, it follows that b can be so chosen that $S(x) \not\equiv 0$. For, if $S(x) \equiv 0$, $C(x + b) = C(x-b)$, that is, $C(2b) = C(0) = 1$. But since $C(x) \not\equiv 1$, there is some value $2b$ of x such that $C(2b) \neq 1$.

If in (3), $x = 0$, $C(- y) = C(y)$. Furthermore

$$\begin{aligned} 2S(- x) &= C(- x - b) - C(- x + b) \\ &= C(x + b) - C(x - b) = - 2S(x). \end{aligned}$$

Now

$$\begin{aligned} S(x + y) - S(x - y) &= \tfrac{1}{2}[C(x + y - b) + C(x - y + b)] \\ &\quad - \tfrac{1}{2}[C(x + y + b) + C(x - y - b)] \\ &= C(x)[C(y - b) - C(y + b)] \\ &= 2C(x)S(y). \end{aligned}$$

Subtract this equation from the equation obtained from it by interchanging x and y. Then

$$S(x - y) = S(x)C(y) - C(x)S(y).$$

Since $S(x) \not\equiv 0$ and $C(x)$ is even, then by Theorem I, § 2,

$S(x)$ and $C(x)$ satisfy equations (1) and (2) simultaneously, where

$$k^2 = \frac{C^2(a) - 1}{S^2(a)}, \qquad S(a) \neq 0.$$

But

$$\begin{aligned} S^2(a) &= \tfrac{1}{4}[C^2(a-b) + C^2(a+b) - 2C(a-b)C(a+b)] \\ &= \tfrac{1}{4}[C(a-b) + C(a+b)]^2 - C(a-b)C(a+b) \\ &= C^2(a)C^2(b) - C^2(a) - C^2(b) + 1 \\ &= [C^2(a) - 1][C^2(b) - 1]. \end{aligned}$$

Therefore

$$k^2 = \frac{1}{C^2(b) - 1} = \frac{2}{C(2b) - 1}.$$

The converse theorem is proved by adding $C(x+y)$ and $C(x-y)$, given by (2), from which it follows that $C(x)$ satisfies (3), whence if $C(x) \not\equiv 0$ it satisfies (4).

§ 4. *The Equation* (5): $S(x+y)S(x-y) = S^2(x) - S^2(y)$.

Discarding the trivial solution $S(x) \equiv 0$, suppose a is a value of x for which $S(a) \neq 0$. Since $S(x)/m$ (where m is any constant different from zero) satisfies equation (5), we may suppose that $S(a) = 1$.

THEOREM. *If* $S(x)$ *satisfies equation* (5), *then* $S(x)$ *and* $C(x) = \frac{1}{2}[S(x+a) - S(x-a)]$ *satisfy equations* (1) *and* (2) *simultaneously, where* $k^2 = C^2(a) - 1$.

Interchange x and y. It follows that $S(x+y)S(y-x) = - S(x+y)S(x-y)$. Replace $x+y$ by a and $x-y$ by x. Then $S(-x) = -S(x)$. Now

$$\begin{aligned} 2C(-x) &= S(-x+a) - S(-x-a) \\ &= -S(x-a) + S(x+a) = 2C(x). \end{aligned}$$

Moreover $C(0) = 1$. From equation (5) it readily follows that

$$S(x+y) = S(x+y)S(a) = S^2\left(\frac{x+y+a}{2}\right) - S^2\left(\frac{x+y-a}{2}\right)$$

and

$$S(x-y) = S(x-y)S(a) = S^2\left(\frac{x-y+a}{2}\right) - S^2\left(\frac{x-y-a}{2}\right).$$

Therefore

$$\begin{aligned} S(x+y)+S(x-y) &= S(x)S(y+a)-S(x)S(y-a) \\ &= 2S(x)C(y). \end{aligned}$$

Interchange x and y and subtract the equation thus obtained from the last equation. Then

$$S(x-y)=S(x)C(y)-C(x)S(y).$$

The hypothesis of Theorem I, § 2, is fulfilled. Since $C(x)$ is even the functions $S(x)$ and $C(x)$ satisfy equations (1) and (2) simultaneously.

To prove the converse theorem when $S(x) \not\equiv 0$, let us observe that if $y = x$ in (2), $C^2(x) - k^2S^2(x) = 1$, whence

$$\begin{aligned} S(x+y)S(x-y) &= S^2(x)C^2(y)-C^2(x)S^2(y) \\ &= S^2(x)-S^2(y), \end{aligned}$$

that is, $S(x)$ satisfies equation (5).

§ 5. *Equation* (6).

THEOREM. *If $g(x)$ and $S(x)$ satisfy equation* (6), *then $S(x)$ satisfies equation* (5) *and $g(x)$ can be so chosen that* (*by the addition of a constant*) $g(2x) = 2k^2S^2(x) + 1$.

If $S(x) \equiv 0$, $g(2x) \equiv g(0)$. Setting aside this trivial solution, it may be assumed that $S(a) \neq 0$. In fact, if $S(a) = m$, a new value of k may be chosen so that $S(a) = 1$. If $x = a$ and $y = -y$, $S(-y) = -S(y)$ and $S(0) = 0$. If $x = 0$, then $g(-y) = g(y)$. Since $\bar{g}(x) = g(x) + \bar{g}(0) - 1$ also satisfies (6), we may assume $g(0) = 1$. If $y = x$, then

$$g(2x)=2k^2S^2(x)+1.$$

Now in (6) replace x by $x + y$ and y by $x - y$. Then

$$S(x+y)S(x-y)=S^2(x)-S^2(y),$$

that is, $S(x)$ satisfies (5).

§ 6. *Equation* (7).

THEOREM. *Suppose the functions $f(x)$ and $C(x)$ satisfy equation* (7) *and $f(x) \not\equiv 0$. If $f(x)$ is not an odd function, its* *even component is a constant multiple of* $C(x)$, *and* $C(x)$ *satisfies equation* (3) *and is not identically zero. If* $f(x)$ *is not an even function,* $C(x)$ *and the odd component* $S(x)$ *of* $f(x)$ *satisfy equations* (1) *and* (2), *where*

$$k^2 = \frac{C^2(\bar{a}) - 1}{S^2(\bar{a})}, \qquad S(\bar{a}) \neq 0.$$

If $f(x) \not\equiv 0$, we may assume that $f(a) = 1$. Let $x = a$ and replace y by $-y$. Then $C(-y) = C(y)$. Let $y = 0$, $x = a$. Then $C(0) = 1$. Hence $C(x) \not\equiv 0$. Let $f(x) = C_1(x) + S(x)$ where $C_1(x)$ is even and $S(x)$ is odd. Then

$$[C_1(x+y) + C_1(x-y)] + [S(x+y) + S(x-y)] = 2C_1(x)C(y) + 2S(x)C(y).$$

Add this equation to the equation obtained from it by replacing x by $-x$ and y by $-y$. Then

$$C_1(x+y) + C_1(x-y) = 2C_1(x)C(y).$$

Interchanging x and y, it is obvious that $C_1(x)C(y) = C_1(y)C(x)$. If $y = 0$, $C_1(x) = k_1C(x)$ where k_1 is a constant. Substituting this value of $C_1(x)$, it is at once apparent that $C(x)$ satisfies (3), if $k_1 \neq 0$. If $k_1 = 0$, $C_1(x) \equiv 0$ and $f(x) \equiv S(x)$. But

$$S(x+y) + S(x-y) = 2S(x)C(y).$$

Interchanging x and y and subtracting the equation thus obtained from the last equation,

$$S(x-y) = S(x)C(y) - C(x)S(y).$$

Therefore, by Theorem I, § 2, $S(x)$ and $C(x)$ satisfy (1) and (2) simultaneously.

§ 7. *Equation* (8).

THEOREM. *If the functions* $S(x)$ *and* $C(x)$ *satisfy equation* (8) *and* $S(x) \not\equiv S(0)$, *they satisfy equations* (1) *and* (2) *simultaneously, where*

$$k^2 = \frac{C^2(a) - 1}{S^2(a)}, \qquad S(a) \neq 0.$$

If $S(x) \not\equiv S(0)$, $C(x) \not\equiv 0$ and there is a value b of x such that $C(b) \neq 0$. Let $x = b$ and replace y by $-y$. Then

$S(-y) = -S(y)$, and $S(0) = 0$. Let $y = a$ and replace x by $-x$, where a is chosen so that $S(a) \neq 0$. Then $C(-x) = C(x)$. If $y = a$ and $x = 0$, $C(0) = 1$. Subtracting equation (8) from the equation obtained from it by interchanging x and y, we have (1), whence by Theorem I, § 2, $S(x)$ and $C(x)$ satisfy (1) and (2) simultaneously.

§ 8. *A General Equation.*

The equation proposed for consideration is

$$(9) \quad \alpha F(x+y)F(x-y) = \beta\varphi(\mu x) + \gamma\psi(\eta y) + \delta, \quad \alpha, \beta, \gamma \neq 0.$$

If $x = y = 0$,

$$\alpha F^2(0) = \beta\varphi(0) + \gamma\psi(0) + \delta.$$

Now if $y = 0$

$$\beta\varphi(\mu x) = \alpha F^2(x) + \beta\varphi(0) - \alpha F^2(0)$$

and if $x = 0$

$$\gamma\psi(\eta y) = \alpha F(y)F(-y) + \gamma\psi(0) - \alpha F^2(0).$$

Therefore

$$F(x+y)F(x-y) = F^2(x) + F(y)F(-y) - F^2(0).$$

CASE I. $F(0) = 0$.

Let $y = x$. Then $F(x)F(-x) = -F^2(x)$ and $F(x)$ satisfies equation (5).

CASE II. $F(0) = k \neq 0$.

Let $F(x) = kC(x) + S(x)$ where $C(x)$ is even and $S(x)$ is odd. Now

$$\begin{aligned} &k^2C(x+y)C(x-y) + S(x+y)S(x-y) \\ &\quad + kC(x+y)S(x-y) + kC(x-y)S(x+y) \\ &\quad = k^2C^2(x) + 2kC(x)S(x) + S^2(x) + k^2C^2(y) - S^2(y) - k^2. \end{aligned}$$

Subtract from this equation the equation obtained by replacing x by $-x$ and y by $-y$. Then

$$C(x+y)S(x-y) + C(x-y)S(x+y) = 2C(x)S(x).$$

Let $y = x$. Then

$$S(2x) = 2C(x)S(x) = C(x+y)S(x-y) + C(x-y)S(x+y).$$

Replace $2x$ by $x - y$ and $2y$ by $x + y$.

$$S(x - y) = S(x)C(y) - C(x)S(y).$$

If y is replaced by $- y$,

$$S(x + y) = S(x)C(y) + C(x)S(y).$$

Therefore

$$\begin{aligned} S^2(x + y) - S^2(x - y) &= 4S(x)C(x)S(y)C(y) \\ &= S(2x)S(2y). \end{aligned}$$

Replace $2x$ by $x + y$ and $2y$ by $x - y$. Then

$$S(x + y)S(x - y) = S^2(x) - S^2(y).$$

Substituting the relations found, it follows that

$$C(x + y)C(x - y) = C^2(x) + C^2(y) - 1,$$

that is, the odd component of $F(x)$ satisfies (5) while the even component (except for the factor $F(0)$) satisfies (4).

STATE UNIVERSITY OF IOWA,
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THE EQUATION $ds^2 = dx^2 + dy^2 + dz^2$.

BY PROFESSOR E. T. BELL.

1. THIS equation,* being of geometrical importance, has attracted several writers, including Serret (1847), Darboux (1873, 1887), de Montcheuil (1905), Salkowski (1909), Eisenhart (1911), and Pell (1918). The simple parametric solution of de Montcheuil, which is the starting point of considerable work in differential geometry, was not noticed by Serret or Darboux. It is somewhat remarkable that the latter overlooked this solution, as he himself makes use (Surfaces,

* Full references to earlier writers are given by Eisenhart, *Annals of Math.* (2), vol. 13 (1911), pp. 17–35. Pell's paper will be found ibid. (2), vol. 20, pp. 142–148. The substance of the present note, with the exception of section 8, is from an unpublished A.M. thesis, presented to the University of Washington in 1908, dealing with the general algebraic problems on which solutions of this kind depend. I wish to emphasize that § 8 was written only after I had read Pell's paper.