imaginary fields, there exist $s = r + c - 1$ units $\epsilon_1, \ldots, \epsilon_s$ in F such that every unit ϵ of F can be expressed in one and but one way in the form

$$\epsilon = \rho\epsilon_1^{a_1} \cdots \epsilon_s^{a_s},$$

where $a_1, \ldots, a_s$ are rational integers, positive, negative or zero, while ρ is a root of unity belonging to the field F. Kronecker* gave a proof along the lines suggested by Dirichlet. Another algebraic proof had been given by Dedekind.†

Minkowski‡ gave a new proof which is largely algebraic, but makes some use of his geometric results. Later, he gave§ a more purely geometric form to his proof, confining his discussion to the typical cases of cubic and quartic fields. This proof makes use of various ideas and results of the geometry of numbers and shows the power and attractiveness of the latter subject.

5. *Classes of Ideals*.—In his proof of the finiteness of the number of classes of ideals in a cubic field, Minkowski|| made use of his results on the upper bound of a product of three linear forms. The process enables us actually to find representatives of the various existing classes of ideals. This problem would be simplified by the discovery of more exact upper bounds.

PRODUCTS OF SKEW-SYMMETRIC MATRICES.

BY PROFESSOR A. A. BENNETT.

IN the March (1919) number of the BULLETIN occurs (page 281) the following: The philosophical faculty of the University of Berlin announces the following prize problem: "To determine by means of the theory of elementary divisors, the criteria that a given matrix be capable of representation as the composition of two skew-symmetric matrices." A dis-

* *Comptes Rendus*, Paris, vol. 96, 1883; vol. 99, 1884; Werke, III_1, pp. 1–30.

† Dirichlet-Dedekind, Zahlentheorie, ed. 2, 1871, § 166, pp. 471–9; ed. 3, 1879, § 177, pp. 555–567; ed. 4, 1894, § 183, pp. 590–603. Cf. Hilbert, *Jahresbericht der Deutschen Math.-Vereinigung*, vol. 4, 1894–5, pp. 214–222.

‡ Geometrie der Zahlen, 1896, pp. 135–147.

§ Diophantische Approximationen, 1907, pp. 133–148.

|| Diophantische Approximationen, pp. 162–167.

cussion of this topic appears therefore of timely interest. To avoid prolixity, only the case of non-singular matrices will be here considered.

For the sake of brevity and clarity the order of the proof will not be interrupted to prove elementary lemmas which in themselves present no difficulty, but several of such lemmas will be stated without proof at the start. Some of these are proved in all discussions of elementary divisors and others while not so familiar present no new difficulties. The definitions and notations employed are those of Bôcher: Introduction to Higher Algebra.

Some Preliminary Lemmas and Definitions.

1. If P is any non-singular matrix, there exist uniquely defined non-singular matrices P' and P^{-1}, known respectively as the conjugate or transposed, and as the inverse of the given matrix. Further, $(P')^{-1} = (P^{-1})'$ and $P^{-1}P = PP^{-1} = I$, where I is the identity, i. e., $MI = IM = M$ for every square matrix M of the same order as I.

2. Skew-symmetric matrices of odd order are necessarily singular, but non-singular skew-symmetric matrices exist of all even orders.

3. If A and B be square matrices of the same order m and with constant elements, the matrix $A + \lambda B$, where λ is variable, is called a linear λ-matrix. The greatest common divisor of all determinants of order j, formed by suppressing $n - j$ rows and $n - j$ columns of $A + \lambda B$, will be a polynomial in λ. If the arbitrary numerical multiplier in the definition of the greatest common divisor be so taken that the non-vanishing term of highest degree in λ has unity for coefficient, this polynomial is denoted by $D_j(\lambda)$. The ratio $D_i(\lambda)/D_{i-1}(\lambda)$, where $D_0(\lambda)$ is arbitrarily defined as unity, is called the ith invariant factor, and is denoted by $E_i(\lambda)$. For i odd, E_i is called an odd invariant factor, and for i even, an even invariant factor.

4. Operations on matrices which merely interchange rows or columns, or multiply rows or columns by non-vanishing factors, or replace a given row by the sum of this row and another, or likewise with columns, are called elementary transformations.

5. If each of two linear λ-matrices is carried into the other by means of elementary transformations, these have the same

invariant factors. Hence, in particular, $PMP^{-1} - \lambda I$ and $M - \lambda I$ have the same invariant factors, for any matrix P of the same order, since the existence of P^{-1} implies that P is non-singular.

6. There exists a standard skew-symmetric matrix T of order $2n$, defined by the condition that if t_{ij} be a general element of T, $t_{ij} = 0$, for $j \neq 2n - i$, $t_{i,\,2n-i} = -1$, $0 < i \leqq n$, $t_{i,\,2n-i} = +1$, $n < i \leqq 2n$. This standard matrix T is called "standard," for the reason that for any non-singular skew-symmetric matrix S of order $2n$, there is a non-singular matrix, P, such that $PSP' = T$.

7. If a matrix K of order $2n$ be such as to have throughout among its elements $a_{ij} = a_{n+i,\,n+j}$, $i \leqq n$, $j \leqq n$, and $a_{ij} = 0$, if $i \leqq n < j$ or $j \leqq n < i$, then the submatrix K_0 consisting of the elements a_{ij}, where $i \leqq n$, $j \leqq n$, determines the invariant factors of K. Indeed if $F_1, F_2, \ldots, F_n$, be the invariant factors of K_0, $E_{2i-1} = E_{2i} = F_i$, $i = 1, 2, \cdots, n$, where E_j is the jth invariant factor of K, $j = 1, 2, \cdots, 2n$.

8. If S_1 and S_2 are any two non-singular skew-symmetric matrices of the same order, $2n$, the linear λ-matrix $S_1 + \lambda S_2$ may be reduced by elementary transformations to a form $L + \lambda T$, where T is the standard matrix defined above, and the elements a of L satisfy the conditions that $a_{i,\,2n-h} = -a_{2n-h,\,i}$, $i \leqq n$, $h \leqq n$, $a_{i,\,2n-h} = 0$, $i \leqq n \leqq h$, or $h < n < i$. Further specialization of the skew-symmetric matrix L is also always possible.

Discussion of the Problem.

Let M be the given non-singular matrix for which it is desired to determine whether M is of the form S_1S_2, where S_1, S_2 are skew-symmetric matrices. By (2), M must be of even order, say $2n$, since otherwise both S_1 and S_2 would be singular and hence M also. One may select a non-singular matrix, P, such that $PMP^{-1} = (PS_1P')(P'^{-1}S_2P^{-1})$ is of the form $M_1 = TS$. The matrix $M_1 - \lambda I$, or $TS - \lambda I$, becomes, on changing the sign of the first n rows and rearranging rows, $S + \lambda T$, the S and T having the same meaning as previously. Using (8), this may be put in the form $L + \lambda T$. Changing the sign of the last n rows and again rearranging rows, this may be written $K - \lambda I$, K being as in (7). From this one has the following conclusion:

THEOREM. *A necessary and sufficient condition that a non-singular matrix, M, shall be expressible as the product of two skew-symmetric matrices, viz., $M = S_1S_2$, is that every even invariant factor of the linear λ-matrix, $M - \lambda I$, shall be equal to the preceding odd invariant factor.**

UNIVERSITY OF TEXAS.

ON THE FIRST FACTOR OF THE CLASS NUMBER OF A CYCLOTOMIC FIELD.

BY MR. H. S. VANDIVER.

(Read before the American Mathematical Society April 27, 1918.)

LET l be an odd prime rational integer and consider the cyclotomic field defined by $e^{2i\pi/l}$. A number of questions connected with this field depend on the divisibility of its class number by l and its powers. This class number can be expressed as the product of two integral factors one of which (generally referred to as the first factor) is

$$(1) \qquad h = \frac{f(Z)f(Z^3) \cdots f(Z^{l-2})}{(2l)^{\frac{1}{2}(l-3)}},$$

where

$$f(x) = r_0 + r_1x + r_2x^2 + \cdots + r_{l-2}x^{l-2},$$

$Z = e^{2i\pi/l-1}$, r is a primitive root of l, and r_i is the least positive residue of r^i, modulo l.

Kummer† proved that the necessary and sufficient condition that h be divisible by l is that one of the numbers of Bernoulli, B_s, $[s = 1, 2, \cdots, (l-3)/2]$ is divisible by l, a B being termed divisible by an integer i when its denominator is prime to i and its numerator is divisible by i. Kronecker‡ gave another proof which was reproduced by Hilbert.§

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* Otherwise expressed, the condition is that the number of integers within parentheses in the characteristic shall always be even, and these alike in pairs. Thus [(2, 2); (3, 3, 1, 1); (1, 1)] is possible, while [(2, 1); (2, 1)]; [2], and [1, 1] are impossible.

† *Journal für die Mathematik*, vol. 40 (1850.)

‡ *Werke*, vol. 1, p. 93.

§ Die Theorie der algebraischen Zahlkörper, Bericht, p. 429.