DETERMINANT GROUPS.

BY PROFESSOR G. A. MILLER.

(Read before the American Mathematical Society April 13, 1918.)

§1. *Introduction.*

LET D represent a determinant of order n whose n^2 elements are regarded as independent variables. The substitutions on these n^2 elements which transform D into itself constitute a substitution group G, which we shall call the *determinant group* of degree n^2. As the elements of D are supposed to be independent variables, it results from the definition of a determinant that every substitution of G must transform the elements of D in such a manner that all the elements of a line (row or column) appear in a line after the transformation.

Hence the substitutions of G correspond to the permutations of the elements of D resulting from transforming its rows and columns independently according to the alternating group of degree n, transforming its rows and columns simultaneously according to negative substitutions in the symmetric group of this degree, and interchanging the rows and columns. The order of G is therefore $(n!)^2$, and hence the number of the distinct determinants that can be formed by permuting the n^2 elements of D is $n^2!/(n!)^2$.* These determinants may be arranged in pairs such that each pair is composed of the determinants which differ only with respect to sign. In particular, the square of D is transformed into itself by a group K whose order is twice the order of G and which contains G as an invariant subgroup.

Some of the abstract properties of G follow directly from the fact that it is simply isomorphic with the imprimitive substitution group of degree $2n$ whose head is composed of the positive substitutions in the direct product of two symmetric groups of degree n. These substitutions correspond to interchanges of the rows among themselves and the columns among themselves or a combination of such interchanges. The re-

* G. Bagnera, *Giornale di Matematiche*, vol. 25 (1887), p. 228; it may be noted that in the review of this article in *Jahrbuch über die Fortschritte der Mathematik* the author's name appears in the form Bergnera.

maining substitutions of this imprimitive group correspond to the interchanges of rows and columns. Hence it results directly that G *contains exactly three subgroups of index* 2 *for every value of* n.

When $n > 4$, G contains no invariant subgroup besides the three of index 2 mentioned above and their common subgroup of index 4 under G, which is also the commutator subgroup of G. In this case G is unsolvable and its factors of composition are 2, 2, $n!/2$, $n!/2$. When $n = 2$, G is simply isomorphic with the four-group, when $n = 3$ it is simply isomorphic with the square of the non-cyclic group of order 6, and when $n = 4$ it contains an invariant abelian subgroup of order 16 and of type (1, 1, 1, 1) which is complementary to a quotient group of order 36 simply isomorphic with the square of the non-cyclic group of order 6.

§ 2. *Determinant Group as a Substitution Group of Degree* n^2.

It has been noted that the substitution group of degree $2n$ which is simply isomorphic with G and corresponds to the permutation of the rows and columns of D is always imprimitive. On the other hand, it is easy to prove that G itself is intransitive when $n = 2$, imprimitive when $n = 3$, and both simply transitive and primitive for every value of n which exceeds 3. In fact, the cases when $n = 2$ or 3 can readily be established by means of the well-known lists of possible substitution groups of degrees 4 and 9.*

That G is always simply transitive when $n > 3$ results directly from the fact that all its substitutions which omit a given letter constitute a subgroup having two transitive constituents: One of these is on the $2n - 2$ other letters in the row and column which contain the fixed letter, while the other transitive constituent is on the $(n - 1)^2$ letters not found in this row or this column. The fact that G is primitive is a consequence of the theorem that *if the subgroup composed of all the substitutions which omit one letter of an imprimitive group omits only one letter, then the degree of one of the transitive constituents of this subgroup increased by one must divide the degree of the imprimitive group.*

The group G contains exactly two substitutions which trans-

* F. N. Cole, *Bulletin of the New York Mathematical Society*, vol. 2 (1893), p. 252.

form each of the elements of any term of D into itself. In particular, the principal term of D is transformed into itself by the substitution of order 2 which interchanges the kth row of D and the kth column, $k = 1, 2, \ldots, n$. Each of the terms of D is therefore transformed into itself by at most $2 \cdot n!$ of the substitutions of G. As such a term cannot be transformed under G into more than $n!/2$ terms, it results that this is the actual number of the substitutions of G which transform a term into itself. The elements of each term of D must therefore be transformed under G according to the symmetric group of degree n.

The positive terms of D, as well as the negative ones, are transformed under G according to a simply transitive group of degree $n!/2$ since the order of G is not divisible by $n!/2 - 1$, whenever $n > 3$. The two substitutions of G which transform each of the elements of one of these terms into itself constitute a group of degree $n(n - 1)$, hence G involves two complete sets of $n!/2$ conjugate substitutions of order 2 and of degree $n(n - 1)$. These sets form a single set of conjugates under the group of D^2 and each of their substitutions is invariant under a group which is simply isomorphic with the direct product of the symmetric group of degree n and a group of order 2. As this group of order $2 \cdot n!$ is a maximal subgroup of G, it results that the simply transitive groups of degree $n!/2$ according to which the positive and the negative terms of D are transformed under G must be primitive whenever $n > 3$.

It was noted above that the subgroup composed of all the substitutions of G which omit a given letter has two transitive constituents of degrees $2(n - 1)$ and $(n - 1)^2$ respectively. This subgroup is evidently formed by a simple isomorphism between these constituent groups and hence each of these groups has for its order the square of $(n - 1)!$. The former constituent is imprimitive, being formed by extending, by a substitution of order 2 which merely interchanges the corresponding letters, the positive substitutions in the direct product of two symmetric groups of degree $n - 1$ written on distinct sets of letters. The latter constituent is the group of the determinant of order $n - 1$ and hence is a simply transitive primitive group, whenever $n > 4$. In particular, *the subgroup of the determinant group of degree* n^2 *which is composed of all its substitutions omitting a given letter is simply isomorphic with the group of the determinant of order* $n - 1$ *and has this group for a transitive constituent whenever* $n > 3$.

Although G is a simply transitive primitive group whenever $n > 3$, it always contains a subgroup of index 2 which is imprimitive, viz., the subgroup corresponding to the interchanges of the rows and columns according to the positive substitutions in the square of the symmetric group. In fact, this subgroup has two sets of systems of imprimitivity, one composed of the elements of rows and the other composed of elements of columns. Each system of the first set has one and only one element in common with each system of the second set. This subgroup contains no other set of systems of imprimitivity and these two sets are transformed into each other under the primitive group G.

The simply transitive primitive group G is evidently of class $3n$, $n > 3$, and hence we have here an interesting infinite system of simply transitive primitive groups in which there are groups for which the ratio of the degree to the class exceeds any given finite number, this ratio being $n/3$. In § 4 we shall consider another infinite system of such groups for which this ratio is still larger. When a primitive group is at least doubly transitive it is well known that this ratio cannot exceed 4 unless the group is either alternating or symmetric.*

§ 3. *Determinant Groups as Substitution Groups of Degree $n!/2$.*

When $n = 2$, D has only one positive term, which is therefore invariant under G. When $n = 3$, D has three positive terms, which are transformed under G according to the symmetric group of degree 3 and therefore each of these three terms is transformed into itself by six of the substitutions of G. These six substitutions transform the three negative terms of D according to the symmetric group of degree 3, hence G contains no substitution besides identity which is commutative with each of the six terms of D.

It has been noted that G transforms the $n!/2$ positive terms of D according to a simply transitive primitive group G', which is simply isomorphic with G whenever $n > 3$ because G involves no invariant subgroup whose index exceeds 4. Let T_1 be the principal diagonal term of D, and let T_2 be a positive term of D which has the last $n - 3$ elements in common with T_1 but is not identical with T_1. Among the $2 \cdot n!$ substitutions of G which transform T_1 into itself there are

* Cf. Encyclopédie des Sciences mathématiques, Tribune publique 14, no. 299.

$6\cdot(n-3)!$ which also transform T_2 into itself. The subgroup of G composed of all its substitutions which omit a given letter must therefore contain a transitive constituent of degree $[n(n-1)(n-2)]/3$.

The number of the transitive constituents of this subgroup varies with n and increases without limit as n increases without limit, since the positive terms which have α elements in common with T_1 can clearly not be transformed under this subgroup into those which have $\beta \neq \alpha$ elements in common with T_1. When $n > 4$ each of these constituents is either of order $n!$ or of twice this order, because this subgroup has only one invariant subgroup whose index exceeds 4 and could not have a transitive constituents of order 2 or of order 4. In fact, each term which has an element in common with T_1 has evidently more than four conjugates under this subgroup whenever $n > 4$, and if a term has no element in common with T_1 its element which occurs in the first row, for instance, can be transformed into $n-1$ other elements of this row under this subgroup.

Among the substitutions which transform T_1 into itself there are $2^{e+1}\cdot e!$ substitutions which also transform the secondary diagonal term of D into itself, e being the largest integer which does not exceed $n/2$. Hence it results that the secondary diagonal term of D is always transformed into an odd number of conjugates under the group formed by the substitutions which transform T_1 into itself, $2\cdot n! \div 2^{e+1}\cdot e!$ being an odd integer as can readily be proved. In particular, when the secondary diagonal term of D is positive the subgroup of G' composed of all its substitutions which omit a given letter must contain a transitive constituent of odd degree and hence of order $n!$ whenever $n > 4$. It may be noted in passing that the given considerations furnish also a proof of the known theorem that the continued product $(m+1)(m+2)\ldots(2m)$ is always divisible by 2^m.

§ 4. *Group of the Square of a Determinant.*

In § 1 it was noted that the substitution group K on the n^2 elements of D composed of all the substitutions on these elements which transform D^2 into itself is of order $2(n!)^2$ and contains G invariantly. Its factors of composition are therefore 2, 2, 2, $n!/2$, $n!/2$. When $n = 2$, it is the octic group and hence it is imprimitive. For all other values of n it is a simply transitive primitive group. In fact, it could not be doubly transitive, since the subgroup composed of all its substitutions which omit a given element has evidently two transitive constituents of degrees $2n - 2$ and $(n - 1)^2$ respectively. Moreover, it could not be imprimitive when n exceeds 3, since it contains the primitive group G which is also of degree n^2. In the special case when $n = 3, K$ is clearly again primitive since its subgroup composed of all its substitutions omitting one letter has two transitive constituents of degree 4, and 9 is not divisible by $4 + 1$.

One of the most interesting facts connected with the infinite system of simply transitive primitive groups represented by K is that these groups are of a very low class. It was noted near the close of § 2 that the class of a primitive group which is at least doubly transitive and does not include the alternating group cannot be less than its degree divided by 4; but when a primitive group is only simply transitive C. Jordan already observed that the ratio of the degree to the class has no upper limit. For the system of simply transitive primitive groups used by him as an illustrative example this ratio becomes however only about one half as large as in the present case since K *is a simply transitive primitive group of degree* n^2 *and of class* $2n$ *whenever* $n > 2$.*

As a transitive substitution group on the rows and columns of D, K is clearly the largest possible imprimitive group of degree $2n$ which has two systems of imprimitivity and hence it includes all the possible imprimitive groups of degree $2n$ which contain two such systems. As a transitive group on the $n!$ terms of D it is also imprimitive when $n > 2$, since it transforms the positive and negative terms of D into each other and involves a subgroup of index 2 which transforms these terms respectively among themselves. When $n > 3$ this subgroup is obtained by establishing a simple isomorphism between two simply transitive primitive groups, when $n = 3$ it is the direct product of two symmetric groups of degree n, and when $n = 2$ it is identity.

Since K contains a subgroup of index 2 which is simply isomorphic with the square of the symmetric group of degree n, it results directly that its smallest invariant subgroup is of

* Cf. W. A. Manning, *American Journal of Mathematics*, vol. 32 (1910), p. 256.

index 8 whenever $n > 4$, and that the corresponding quotient group is the octic group. Hence K involves exactly three subgroups of index 2 whenever n exceeds 4 and only two other invariants subgroups besides identity, viz., the mentioned subgroup of index 8 and one of index 4 corresponding to the invariant subgroup of order 2 of the octic group. These results apply also to the special case when $n = 3$.

TRANSLATION SURFACES IN HYPERSPACE.

BY PROFESSOR C. L. E. MOORE.

(Read before the American Mathematical Society, April 27, 1918.)

1. If the rectangular coordinates of the points of a surface can be expressed in the parametric form

$$x_i = f_i(u) + g_i(v) \qquad (i = 1, 2, \ldots, n), \tag{1}$$

where f_i are functions of u alone and g_i functions of v alone, the surface is called a translation surface. It is seen that a translation can be found which will send any parameter curve $u =$ const. into any other one of the same system. The same is true of the curves $v =$ const. The surface (1) is also seen to be the locus of the mid-points of the lines joining the points of

$$C_1: \quad x_i = 2g_i(u) \quad \text{to the points of} \quad C_2: \quad x_i = 2f_i(v). \tag{2}$$

The character of the surface can then be determined, in a great measure, by the form and relative position of these two curves. Nearly all writers on surface theory* mention three facts concerning translation surfaces in 3-space:

(a) The generators of the developable which touches the surface along a curve $u =$ const. are tangent to the curves $v =$ const., or in other words the directions of the parameter curves passing through a given point are conjugate directions.

(b) There are surfaces which can be expressed in more than one way in the form (1).

* Darboux, Théorie générale des Surfaces, vol. 1, pp. 148, 340. Scheffers, Theorie der Flächen, vol. 2, pp. 188, 245.