*3 sextactic points are contacts of tangents from the flexes* $P_3$. *The* 6 *contacts of tangents from the sextactic points are the points* $P_{12}$. *The* 12 *contacts of tangents from* $P_{12}$ *in turn are the points* $P_{24}$, *and so on ad infinitum.*

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# RELATED INVARIANTS OF TWO RATIONAL SEXTICS.

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LET the parametric equations of the $R_3{}^6$, the rational curve of order six in three dimensions, be

$$x_i = \delta^6{}_{it} \equiv a_i t^6 + 6b_i t^5 + 15c_i t^4 + 20d_i t^3 + 15e_i t^2 + 6f_i t + g_i \quad (i = 1, 2, 3, 4), \tag{1}$$

and let the parametric equations of the $R_2{}^6$, the rational plane curve of order six, be of the form

$$\begin{aligned} x_1 &= \alpha_t{}^6 \equiv a + bt + ct^2 + dt^3 + et^4 + ft^5 + gt^6, \\ x_2 &= \beta_t{}^6 \equiv a' + b't + c't^2 + d't^3 + e't^4 + f't^5 + g't^6, \\ x_3 &= \gamma_t{}^6 \equiv a'' + b''t + c''t^2 + d''t^3 + e''t^4 + f''t^5 + g''t^6. \end{aligned}$$

It is well known that all plane sections of the $R_3{}^6$ are apolar to a doubly infinite system of binary sextics, and that all line sections of the $R_2{}^6$ are apolar to a triply infinite system of binary sextics. We shall let the four binary sextics $\delta_{it}{}^6$ of (1) be four linearly independent sextics of the apolar system of the $R_2{}^6$, and the $\alpha_t{}^6, \beta_t{}^6, \gamma_t{}^6$ of (2) be three linearly independent sextics of the apolar system of the $R_3{}^6$. Our purpose is to point out briefly the relation between the invariants of the $R_2{}^6$ and the invariants* of the $R_3{}^6$.

By means of the twelve equations

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* This relation must not be confused with the correspondence between invariants of the $R_2{}^n$ and covariant surfaces of the $R_3{}^n$.

$$
\begin{aligned}
a_ia - b_ib + c_ic - d_id + e_ie - f_if + g_ig &= 0,\\
a_ia' - b_ib' + c_ic' - d_id' + e_ie' - f_if' + g_ig' &= 0,\\
a_ia'' - b_ib'' + c_ic'' - d_id'' + e_ie'' - f_if'' + g_ig'' &= 0
\end{aligned}
\tag{3}
$$

$$(i = 1, 2, 3, 4),$$

it may be easily proved that the four-rowed determinants of the matrix of the coefficients of $\delta_{it}^6$ of the type $|abcd|$ are proportional to the complementary three-rowed determinants of the matrix of the coefficients of $\alpha_t^6$, $\beta_t^6$, $\gamma_t^6$ of the type $|ef'g''|$. Let $T$ denote the substitution of the three-rowed determinants of (2) for the proportional four-rowed determinants of (1), and $T^{-1}$ the inverse substitution.

Invariants of the $R_3^6$ are combinants of the four sextics $\delta_{it}^6$, and conversely, and these are rationally expressible in terms of the determinants of the type $|abcd|$. Invariants of the $R_2^6$ are combinants of $\alpha_t^6$, $\beta_t^6$, $\gamma_t^6$, and conversely, and these are rationally expressible in terms of the determinants of the type $|ab'c''|$. The combinants of $\delta_{it}^6$ are implicit invariants of the $R_2^6$ which become explicit invariants of the $R_2^6$ after the application of $T$. Similarly, combinants of $\alpha_t^6$, $\beta_t^6$, $\gamma_t^6$ are implicit invariants of the $R_3^6$ which are transformed into explicit invariants of the $R_3^6$ by means of $T^{-1}$. Hence any explicit invariant $I$ of the $R_3^6$ is transformed into an explicit invariant $I'$ of the $R_2^6$ by means of $T$. Similarly, $T^{-1}\,I' = I$. It is evident that the order of $I$ in the $|abcd|$ is the same as that of $I'$ in the $|ab'c''|$. We shall now mention a few illustrations of this relation.

If $U'$ is the undulation invariant of the $R_2^6$, $T^{-1}\,U' = U$ is the stationary line invariant of the $R_3^6$. From $P$, the pentatactic plane invariant of the $R_3^6$, we obtain $TP = P'$, the cusp invariant of the $R_2^6$. Similarly, from $Q$, the quinquesecant line invariant of the $R_3^6$, we derive $TQ = Q'$ whose vanishing defines an $R_2^6$ such that any six of its collinear points have parameters apolar to a binary quintic. If $N = 0$ is the necessary and sufficient condition that the $R_3^6$ have a node, $TN = N' = 0$ defines an $R_2^6$ which has one secant that cuts out a cyclotomic set of parameters.

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