INVOLUTIONS ON THE RATIONAL CUBIC.

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(Read before the San Francisco Section of the American Mathematical Society October 27, 1917.)

Introduction.

1. THE general subject of involution as applied to rational curves has been widely studied, notably by Weyr, Stahl, Coble and many Italian writers. It is the purpose of this paper to discuss certain involutions on the rational cubic, R^3.

If s_i denote the elementary symmetric functions of coordinates $x_1, x_2, \ldots, x_n$ of n points (elements in the binary domain), the most general involution of order n, $I_{n-1.1}$, i. e., one in which $n - 1$ points of a set determine the remaining one, will be defined by

$$a_0 s_n + a_1 s_{n-1} + a_2 s_{n-2} + \cdots + a_{n-1} s_1 + a_n = 0. \tag{1}$$

The involution is thus made up of all sets of n points apolar to a fixed set, the n-fold points of the involution, given by

$$\begin{aligned} a_0 x^n + \binom{n}{1} a_1 x^{n-1} + \binom{n}{2} a_2 x^{n-2} + \cdots \\ + \binom{n}{n-1} a_{n-1} x + a_n = 0. \end{aligned} \tag{2}$$

The following alternative and equivalent definition is serviceable when the n points of a set are represented implicitly by an equation: An $I_{n-1,1}$ is an $(n-1)$-parameter family of binary forms of order n

$$f_0 + k_1 f_1 + k_2 f_2 + \cdots + k_{n-1} f_{n-1}. \tag{3}$$

More generally, if $n - r$ points of a set suffice to determine the remaining r, x_i must satisfy r equations of the type (1) and (3) reduces to an $(n-r)$-parameter family. The corresponding involution is denoted by $I_{n-r,\,r}$.

2. Choosing for triangle of reference the nodal tangents and the line of flexes, the equation of the curve may be written in the canonical form

$$x_1 = 3t^2, \qquad x_2 = 3t, \qquad x_3 = t^3 + 1. \tag{4}$$

The peculiar adaptability of the language of involution to describe the properties of the curve will appear from a few examples. Thus the condition that three points be on a line is

$$(5) \qquad s_3 + 1 = 0,$$

i. e., (5) defines the involution which comprises all line sections. If the line is tangent at τ, meeting again at t, the equation becomes

$$(6) \qquad \tau^2 t + 1 = 0.$$

Again (from (6)) *the parameters of contacts of tangents from an arbitrary point of* R^3 *belong to the quadratic involution* $s_1 = 0$ *whose double points are the nodal parameters.**

Finally, *lines joining pairs of points in a quadratic involution envelop a conic perspective to* R^3.†

The line $\overline{t_1 t_2}$ is

$$(7) \qquad (s_1 s_2 - 1)x_1 + (s_1 - s_2^2)x_2 - 3s_2 x_3 = 0.$$

If the double points of the involution are given by

$$(8) \qquad at^2 + 2bt + c = 0,$$

the equations of the perspective conics in lines, found by requiring that (8) be apolar to (7) considered as a quadratic in t_2, are

$$(9) \quad u_1 = bt^2 - ct + a, \qquad u_2 = ct^2 - at + b, \qquad u_3 = -3bt.$$

These conics have each three contacts with R^3 *which belong to an* $I_{2,1}$, $s_3 = 1$, *whose triple points are the sextactic points.*‡

Among the tri-tangent conics must be counted the degenerate conics consisting of two tangents from a point of the curve. *Thus the point* t *and the contacts of tangents from* t *are a set apolar to the sextactic points.* The equation of any composite conic, e. g., the pair of tangents from t_1, is found at once by taking the discriminant of (7) considered as a quadratic in t_2.

Since the triple points are a set in the involution, there is one conic touching at the sextactic points. This is a remarkable conic which we shall call N. It is obtained by requiring

*This theorem, discovered independently, is referred to by Weyr, *Wiener Berichte*, vol. 79 (1879), p. 429 ff., as known.

† Coble, "Symmetric binary forms and involutions," *Amer. Jour. Math.*, vol. 32, p. 358.

‡ The equations of the three sextactic conics are given by (9) when $b = 1$; $a = c = -3, -3\omega, -3\omega^2$, $(\omega^3 = 1)$.

that the double points (8) of the quadratic involution be the nodal parameters. In this case $a = c = 0$ and the conic is

$$(10)\qquad \begin{aligned} &u_1 = t^2,\ u_2 = 1,\ \ u_3 = -3t, \quad \text{or} \quad u_3^2 - 9u_1u_2 = 0,\\ &x_1 = 3,\ x_2 = 3t^2,\ x_3 = 2t, \quad \text{or} \quad 9x_3^2 - 4x_1x_2 = 0. \end{aligned}$$

Combining the foregoing discussion with a theorem stated at the beginning of this section, we may say: *The lines joining pairs of contacts of tangents from the points of R^3 envelop a conic N which touches the nodal tangents (where they meet the line of flexes) and has contacts with R^3 at the sextactic points.*

3. As an example of an $I_{1,2}$ may be mentioned the involution set up by the pencil of lines $(ux) + \lambda (vx) = 0$. The lines will cut out a pencil, i. e., an $I_{1,2}$ of binary cubics, say $u + \lambda v$. The contacts of tangents from the center of the pencil which are the double points of the involution are given by the Jacobian J of u and v.*

Are there any lines u whose cubi-covariant points are also line sections u'? If so, these lines may be taken as the base lines of a pencil and the involution becomes $u + \lambda u'$. That is, from the intersection P of u and u' can be drawn not four tangents but a pair of repeated tangents. This can happen if and only if P is the intersection of two flex tangents. Hence if u is a line of a pencil with center P, its cubi-covariant points lie on a line u' of the same pencil. *In other words the locus of lines u whose cubi-covariant points lie on a line u' consists of the vertices of the triangle of flex tangents, while u' envelops the same points.*

Lines u and u' in any pencil P belong to a quadratic involution of lines whose double lines are the two flex tangents meeting at P.

The points P are $(1, 1, 1)$, $(1, \omega, \omega^2)$, $(1, \omega^2, \omega)$ and are therefore fully perspective with the reference triangle.

Other Contact Conics.

4. To find the intersections of R^3 with a general conic we substitute equations (4) in the trilinear equation of the conic. The result will be a sextic in t, in which obviously the coefficient of the highest power is the same as the constant term. Hence the necessary and sufficient condition that six points lie on a conic is that their parameters satisfy the equation

* Salmon, Higher Algebra, fourth edition, p. 162.

$$(11) \qquad s_6 = 1,$$

or that they belong to a sextic involution $I_{5,1}$.

We shall get contacts when two or more t's come together.* Passing to the extreme case, suppose all six t's coincide. Then

$$(12) \qquad t^6 - 1 = (t^3 + 1)(t^3 - 1) = 0,$$

which includes the flexes among the sextactic points. The conics then degenerate of course into the flex tangents repeated.

5. Next suppose five points coincide at τ. Then

$$(13) \qquad \tau^5 t - 1 = 0,$$

which says that at each point τ of R^3 there is a conic with 5-point contact,† but through a given point t five such conics pass. Moreover the product s_5 of the five quintactic parameters satisfies the equation $s_5 = 1/t$ (from (13)). Hence, by (11), *the quintactic points of the five quintactic conics which pass through* t *(simply) lie on a conic with* t.

6. Let four intersections coincide at τ and two at t, or

$$(14) \qquad \tau^4 t^2 = 1.$$

There would seem to be four conics with simple contact at t and with 4-point contact elsewhere. But among these are the tangent lines from t each counted twice and therefore only two proper conics with contacts satisfying the equation

$$(15) \qquad \tau^2 t = 1.$$

Or the quartactic points τ of conics touching at t are given by

$$(16) \qquad \tau^2 - 1/t = 0.$$

Hence they lie on a line with t. Moreover they are harmonic with the contacts of tangents from t. They are likewise harmonic with the nodal parameters; hence the line joining them is a line of conic N and their tangents meet again on the curve, viz., at $-t$. Again $-t$ is on a line with the contacts of tangents from t. These statements apply equally well to $-t$.

* Contact here simply means coincident intersections and will include improper contact as well as ordinary tangency, i. e., coincidence of consecutive parameters.

† We shall call this a *quintactic* point and the conic a quintactic conic, adopting similar terms for the other cases.

We thus have a striking configuration which may be described as follows:

From a point $(-1/t^2)$ *of* R^3 *draw the two tangents* t *and* $-t$ *and from each of these points the pair of tangents. The two pairs of contacts of tangents from* t *and* $-t$ *are harmonic and form therefore with the nodal parameters three mutually harmonic pairs. The line joining contacts of tangents from* t *passes through* $-t$ *and is therefore a tangent from* $-t$ *to conic* N, *and vice versa. The other tangent to* N *from either point is the junction of the two. At either point* (t *or* $-t$) *two conics touch which have quartactic points at contacts of tangents from the other.*

7. If now the six intersections coincide in two triples, the equation becomes

$$(17) \qquad t^3\tau^3 = 1, \qquad \text{or} \qquad \tau^3 - 1/t^3 = 0.$$

Hence at each point t of R^3 there are three osculating conics which osculate the curve again at points $1/t$, ω/t, ω^2/t respectively. At each of these points there are three osculating conics which osculate again, one each at t, ωt and $\omega^2 t$. *We have thus a configuration of six points and nine conics. If the points are arranged in two rows, as*

$$\begin{array}{ccc} t & \omega t & \omega^2 t \\ 1/t & \omega/t & \omega^2/t \end{array}$$

then at each point in either row three conics osculate, each of which osculates again at one point of the other row. The six points themselves lie on a tenth conic whose equation is

$$9t^3x_3^2 - (t^3 + 1)^2\, x_1x_2 = 0.$$

Moreover the two triangles as written are harmonically perspective from each of the flexes.

8. Finally we obtain tri-tangent conics when the points of (11) coincide in pairs. The contacts then satisfy the equation

$$s_3^2 = t_1^2\, t_2^2\, t_3^2 = 1.$$

But if the right side is -1, the conic is any line section repeated. The contacts of all tri-tangent conics (except repeated lines) therefore belong to the $I_{2,1}$, $s_3 = 1$. Since this is the same involution as that defined by the sextactic points we infer that the perspective conics (9) Art. 2 embrace all tri-tangent conics.

Hyperosculating Curves.

9. The foregoing method can be applied to the study of other contact curves of the R^3. We shall restrict our attention, however, to those which have complete intersection at a point. Such curves of order n will be called *hyperosculating* curves, designated by H_n. The contacts, denoted by P_{3n}, will be called hyperosculating points. The simplest of these curves are the flex tangents, of course, which taken n times must be reckoned among all H_n's. Similarly the sextactic conics counted n times will be included among the H_{2n}'s of even degree, etc.

If now the x's from (4) are substituted in a general ternary equation of degree n, there results a binary equation of degree $3n$ in t, wherein the coefficient of the highest power of t differs at most in sign from the constant term. Denoting by s_{3n} the product of the roots of this equation, we have as the condition that $3n$ points be the complete intersections of R^3 and a curve of degree n

$$s_{3n} = (-1)^n. \tag{18}$$

i. e., that they belong to an involution, $I_{3n-1,1}$.

Hence the contacts, P_{3n}, of hyperosculating curves are given by

$$t^{3n} = (-1)^n. \tag{19}$$

10. There will be two cases according as n is odd or even.

Case I. n odd. When $n = 1$ we have the points of inflexion. That is, there are 3 H_1's whose P_3's lie on a line.

When $n = 3$, (19) is

$$t^9 + 1 = (t^3 + 1)(t^6 - t^3 + 1). \tag{20}$$

There are thus 9 H_3's whose P_9's are on a cubic. But three of these P_9's are the flexes, counted three times, and are therefore the complete intersections of a line and R^3. *Hence there are* 6 *proper H_3's whose contacts are on a conic.*

In general, n odd, the contacts of H_n's are given by

$$t^{3n} + 1 = 0. \tag{21}$$

Here $s_{3n} = -1$. Hence there are $3n$ points at which H_n's, including degenerate cases, can be drawn. These points are the complete intersections of R^3 and a C_n. This C_n, however, is always composite, since it contains the line C_1 of flexes. It may contain also other factors C_{k_i} at whose intersections can be drawn degenerate H_n's which are $H_{n/r}$'s of lower order repeated r times.

If $C_n = C_1 C_{k_1} C_{k_2} \ldots C_{k_j} C_{n'}$, it follows that *there are* $3n'$ P_{3n}*'s at which proper* H_n*'s can be drawn and these points are the complete intersections of* R^3 *and* $C_{n'}$. For the lower values of n the C_n's are appended, from which can be inferred the number of P_{3n}'s at which proper H_n's can be drawn; e. g., there are 12 P_{15}'s at which proper H_5's can be drawn and these points are on a quartic C_4.

$$\begin{gathered} C_3 = C_1C_2, \quad C_5 = C_1C_4, \quad C_7 = C_1C_6, \quad C_9 = C_1C_2C_6, \\ C_{11} = C_1C_{10}, \quad C_{13} = C_1C_{12}, \quad C_{15} = C_1C_2C_4C_8. \end{gathered} \tag{22}$$

11. *Case* II. n even and equal to $2m$. The hyperosculating points are now given by

$$t^{6m} - 1 = (t^{3m} - 1)\,(t^{3m} + 1). \tag{23}$$

There are two cases according as m is odd or even.

(a) When m is odd the second factor of (23) gives points P_{6m} which are P_{3m} taken twice; while the other factor

$$t^{3m} - 1 = (t^3 - 1)\,(t^{3m-3} + t^{3m-6} + \ldots + t^3 + 1). \tag{24}$$

The first factor of (24) corresponds to the sextactic points which are to be taken m times. *The other factor indicates that there are* $(3m - 3)$ P_{3n}*'s which lie on a* C_{m-1}.

(b) When m is even the first factor of (23) names contacts of $(H_m\text{'s})^2$. *The other factor says that there are* $3m$ P_{3n}*'s which lie on a* C_m. These curves C_{m-1} and C_m may or may not be composite, but as above we can infer that the points at which proper H_n's can be drawn are the complete intersections of R^3 and a $C_{m'}$.

In particular if n is a power of 2, say $n = 2^a$, (19) becomes

$$(t^{3\cdot 2^a} - 1) = (t^{3\cdot 2^{a-1}} - 1)\,(t^{3\cdot 2^{a-1}} + 1) = 0, \tag{25}$$

the first factor of which names contacts of curves $H_{2^{a-1}}$ repeated, except when $a = 1$. The second factor gives proper $P_{3\cdot 2^a}$'s which by (18) lie on a $C_{2^{a-1}}$. Moreover in virtue of relation (6) these $P_{3\cdot 2^a}$'s are contacts of tangents from proper $P_{3\cdot 2^{a-1}}$'s. We have thus the following chain theorem: *The*

3 *sextactic points are contacts of tangents from the flexes* P_3. *The* 6 *contacts of tangents from the sextactic points are the points* P_{12}. *The* 12 *contacts of tangents from* P_{12} *in turn are the points* P_{24}, *and so on ad infinitum.*

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RELATED INVARIANTS OF TWO RATIONAL SEXTICS.

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(Read before the American Mathematical Society September 4, 1918.)

LET the parametric equations of the R_3^6, the rational curve of order six in three dimensions, be

$$(1)\qquad x_i = \delta^6{}_{it} \equiv a_it^6 + 6b_it^5 + 15c_it^4 + 20d_it^3 + 15e_it^2 + 6f_it + g_i \quad (i = 1, 2, 3, 4),$$

and let the parametric equations of the R_2^6, the rational plane curve of order six, be of the form

$$\begin{aligned} x_1 &= \alpha_t{}^6 \equiv a + bt + ct^2 + dt^3 + et^4 + ft^5 + gt^6,\\ x_2 &= \beta_t{}^6 \equiv a' + b't + c't^2 + d't^3 + e't^4 + f't^5 + g't^6,\\ x_3 &= \gamma_t{}^6 \equiv a'' + b''t + c''t^2 + d''t^3 + e''t^4 + f''t^5 + g''t^6. \end{aligned}$$

It is well known that all plane sections of the R_3^6 are apolar to a doubly infinite system of binary sextics, and that all line sections of the R_2^6 are apolar to a triply infinite system of binary sextics. We shall let the four binary sextics $\delta_{it}{}^6$ of (1) be four linearly independent sextics of the apolar system of the R_2^6, and the $\alpha_t{}^6, \beta_t{}^6, \gamma_t{}^6$ of (2) be three linearly independent sextics of the apolar system of the R_3^6. Our purpose is to point out briefly the relation between the invariants of the R_2^6 and the invariants* of the R_3^6.

By means of the twelve equations

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* This relation must not be confused with the correspondence between invariants of the R_2^n and covariant surfaces of the R_3^n.