

analytic in (x_0, y_0, z_0) and r_1 and r_2 are not both zero there. These solutions of (3) are, then, analytic except perhaps in points of singularity of c_1, c_2 and in points for which $r_1 = r_2 = 0$. But they are identical with certain solutions of the equations (2), solved for $\partial k_1/\partial y, \partial k_2/\partial y$,—solutions which are analytic except perhaps in points of singularity of c_1, c_2 and in points for which $q_1 = q_2 = 0$. Evidently, then, these solutions k_1, k_2 , and hence the vectors $\gamma_1 = k_1c_1 + k_2c_2, \gamma_2 = k_2c_1 - k_1c_2$, are analytic except perhaps in points of singularity of c_1, c_2 and in points in which both c_1 and c_2 are indeterminate, that is, have all three components zero. Thus we have the theorem:

If the gradients c_1, c_2 of the functions F_1, F_2 in $F = F_1 + iF_2$, where $F_1 = F_1 | [L] |, F_2 = F_2 | [L] |$ are functions of the first degree of the space curve L , are in general analytic, the gradients γ_1, γ_2 of Φ_1, Φ_2 in $\Phi = \Phi_1 + i\Phi_2$, an arbitrary complex function of L of the first degree isogenous to F , are analytic save perhaps in points of singularity of c_1 or c_2 and points in which both these vectors are indeterminate.

The theorem still holds when the vectors c_1, c_2 are proportional. In this case k_1 and k_2 are both solutions of the equation

$$p_i \frac{\partial k}{\partial x} + q_i \frac{\partial k}{\partial y} + r_i \frac{\partial k}{\partial z} = 0, \quad (i = 1, 2),$$

to which both of the equations (3) reduce in form, and γ_1 and γ_2 are both proportional to c_1 and c_2 .

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AN ELEMENTARY DERIVATION OF THE PROBABILITY FUNCTION.

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WE shall derive by means of elementary considerations the equation of the probability curve from the sequence of binomial coefficients. If the asymptotic form of $x!$ be obtained, the problem is very simple but none the less merits attention. The asymptotic form of $n!$, viz., $\sqrt{2\pi n}(n/e)^n e^{\theta/(12n)}, 0 < \theta < 1$, might of course be taken for granted, but so far as is known

to the writer, no purely algebraic attempt to prove Stirling's formula has been successful. The proof here given does use ratios of factorial expressions but proves the asymptotic values of these directly. The original derivation by Gauss is elegant and elementary but does not attempt to show the relation of probability in the finite cases to the transcendental probability function $ce^{-h^2x^2}$ nor does that derivation appear to adapt itself to any modification which will render obvious the fact that the infinite case is but a limit obtained from the finite case.

It is true that the method here described makes use of the convergence and the value approached of a single infinite product, viz., Wallis's formula for $\pi/2$,

$$\frac{\pi}{2} = \frac{2}{1} \cdot \frac{2}{3} \cdot \frac{4}{3} \cdot \frac{4}{5} \cdot \frac{6}{5} \cdot \frac{6}{7} \cdot \frac{8}{7} \cdot \frac{8}{9} \cdots,$$

but this infinite product is a familiar one and is discussed in nearly every textbook treatment of infinite products.

Let a succession of adjacent rectangles R_k with altitudes, $\binom{2m}{m+k} a_m$, and common length of base, b_m , be erected on a straight line. The total area of these rectangles for a given m , with k ranging, is, by the binomial theorem, $2^{2m} a_m b_m$. If a_m and b_m be so varied with m that the middle rectangle R_0 remains of finite height and the area remains finite, then the rectangles form approximations to the probability curve. For the mid-ordinate $\binom{2m}{m} a_m$ to be finite, a_m must be asymptotically proportional to the reciprocal of the middle term $\binom{2m}{m}$.

We seek therefore the asymptotic form of the middle term M_m . Let $C_m = m M_m^2 / 16^m$; then $C_{m+1} / C_m = (2m+1)^2 / 2m(2m+2)$. Now $C_1 = \frac{1}{4}$, hence, in general, for $m > 1$,

$$C_m = \frac{1}{2} \left[\frac{1}{2} \cdot \frac{3}{2} \cdot \frac{3}{4} \cdot \frac{5}{4} \cdot \frac{5}{6} \cdots \frac{(2m-1)^2}{(2m-2)2m} \right].$$

Hence $M_m \sim 4^m / \sqrt{m\pi}$, and apart from an arbitrary constant factor, $a_m \sim \sqrt{m\pi} / 4^m$. Since the area $2^{2m} a_m b_m$ is also to be finite, we have, except for an arbitrary constant factor, $b_m \sim 1 / \sqrt{m}$.

If a definite point x be chosen upon the base line, this point will fall within the rectangle numbered k only if kb_m does not differ from x by as much as b_m ; the k denoting a fixed x must therefore vary with m , and we have $x \sim kb_m$, or $k \sim x \sqrt{m}$.

The asymptotic ratio of the general term $\binom{2m}{m+k}$ to the middle term $\binom{2m}{m}$ may be written in the form

$$\binom{2m}{m+k} / \binom{2m}{m} = \frac{1}{1 + \frac{k}{m}} \cdot \frac{1}{1 + \frac{k}{m-1}} \cdots \frac{1}{1 + \frac{k}{m-k+1}}$$

$$\therefore \left[\frac{1}{1 + \frac{k}{m}} \right]^k > \binom{2m}{m+k} / \binom{2m}{m} > \left[\frac{1}{1 + \frac{k}{m-k+1}} \right]^k.$$

Replacing m partially in accordance with $k \sim x \sqrt{m}$, we have

$$\left(\frac{1}{1 + x^2/k} \right)^k > \binom{2m}{m+k} / \binom{2m}{m} > \left(\frac{1}{1 + x^2/(k - x^2 + x^2/k)} \right)^k.$$

Using the fact that the reciprocal of the middle term is asymptotically equal to the altitude factor a_m , we obtain by passing to the limit for m and hence also for k the asymptotic form of the height of the rectangle covering or touching the point x , viz., $y(x) = e^{-x^2}$.

We have thus obtained the equation $y = e^{-x^2}$ as the limit of the altitude of a rectangle contained in a sequence erected proportional to the binomial coefficients, and by different units of proportionality we would have obtained the form $y(x) = ce^{-h^2x^2}$, geometrically no more general. We have incidentally obtained the area $\int_{-\infty}^{+\infty} e^{-x^2} dx = \lim 4^m a_m b_m = \sqrt{\pi}$, directly, without recourse to such devices as polar integration of a double integral. Even the binomial theorem may be interpreted in the limit, giving

$$f(x \sqrt{2}) = \sqrt{\pi/2} \int_{-\infty}^{+\infty} f(x+t)f(x-t) dt,$$

where $f(t)$ is e^{-x^2} .

This treatment is believed to be original, but the literature available for examination by the author is that customary to an army post, "somewhere on the Gulf of Mexico,"—nil.