THE APRIL MEETING OF THE AMERICAN MATHEMATICAL SOCIETY IN NEW YORK.

THE one hundred and ninety-eighth regular meeting of the Society was held in New York City on Saturday, April 27, 1918, extending through the usual morning and afternoon sessions. The attendance included the following thirty-three members:

Professor R. C. Archibald, Dr. F. W. Beal, Professor F. N. Cole, Professor Elizabeth B. Cowley, Professor Louise D. Cummings, Professor D. R. Curtiss, Professor L. P. Eisenhart, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor O. E. Glenn, Dr. Olive C. Hazlett, Dr. T. R. Hollcroft, Professor Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. J. R. Kline, Professor P. H. Linehan, Professor R. B. McClenon, Professor L. C. Mathewson, Professor R. L. Moore, Professor F. M. Morgan, Professor Anna J. Pell, Dr. J. F. Ritt, Dr. Caroline E. Seely, Professor D. E. Smith, Professor P. F. Smith, Professor H. D. Thompson, Mr. H. S. Vandiver, Professor E. B. Van Vleck, Professor H. S. White, Mr. J. K. Whittemore.

Ex-President H. S. White presided at the morning session and Professor W. B. Fite at the afternoon session. The Council announced the election of the following persons to membership in the Society: Mr. O. S. Adams, U. S. Coast Survey; Professor W. P. Parker, Union Christian College, Pyeng Yang, Korea; Dr. E. F. Simonds, University of Illinois. Seven applications for membership in the Society were received.

Professor P. F. Smith was reelected a member of the Editorial Committee of the *Transactions*. Committees were appointed to consider the question of the publication of the recent Chicago Symposium, and to submit to the Council at the October meeting a list of nominations of officers to be elected at the annual meeting.

The following papers were read at the April meeting:

(1) Professor ARNOLD EMCH: "On plane algebraic curves with a given system of foci."

(2) Dr. J. F. RITT: "On the iteration of polynomials."

(3) Professor F. F. DECKER: "On the order of the system of equations arising from the vanishing of determinants of a given matrix."

(4) Professor O. E. GLENN: "Modular concomitant scales, with a fundamental system of formal covariants, modulo 3, of the binary quadratic."

(5) Professor J. E. ROWE: "The quinquesecant line invariant of the rational sextic curve in space."

(6) Professor F. H. SAFFORD: "Parametric equations of the path of a projectile when the air resistance varies as the nth power of the velocity."

(7) Professor C. L. E. MOORE: "Surfaces of rotation in space of four dimensions."

(8) Professor C. L. E. MOORE: "Translation surfaces in hyperspace."

(9) Dr. MARY F. CURTIS: "Note on the rectifiability of a space cubic."

(10) Professors F. R. SHARPE and VIRGIL SNYDER: "Certain types of involutorial space transformations."

(11) Dr. CAROLINE E. SEELY: "On kernels of positive type."

(12) Lieut. J. W. HOPKINS: "Some convergent developments associated with irregular boundary conditions."

(13) Dr. J. R. KLINE: "A necessary and sufficient condition that a closed connected point set that divides the plane into two domains be a simple curve."

(14) Professor EDWARD KASNER: "Equilong symmetries and a related group."

(15) Professor H. B. PHILLIPS: "Functions of matrices."

(16) Mr. G. H. HALLETT, JR.: "Linear order in three-dimensional euclidean and double elliptic spaces."

(17) Mr. H. S. VANDIVER: "On transformations of the Kummer criteria in connection with Fermat's last theorem."

(18) Mr. H. S. VANDIVER: "A property of cyclotomic integers and its relation to Fermat's last theorem (third paper)."

(19) Mr. H. S. VANDIVER: "On the first factor of the class number of a cyclotomic field."

(20) Mr. H. S. VANDIVER: "Proof of a property of the norm of a cyclotomic integer."

Mr. Hopkins's paper was communicated to the Society and read by Professor Dunham Jackson. Mr. Hallett was introduced by Professor R. L. Moore. In the absence of the authors, the papers of Professor Emch, Professor Decker, Professor Rowe, Professor Safford, Professor C. L. E. Moore, Dr. Curtis, Professors Sharpe and Snyder, Professor Phillips, and the last two papers of Mr. Vandiver were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The methods of Plücker, Siebeck, and others for determining the foci of given algebraic curves are well known. In this paper Professor Emch assumes conversely a given set of foci and determines the system of algebraic curves with the given set of foci. As an application of the method, among other problems, the equation

$$X^{n-2k}\, y^{2k} = (-1)^{n-2k}\frac{(2k)^{2k}}{n^n\,(n-2k)^{2k-n}}$$

of a special class of n-ics, with the nth roots of unity interpreted in the cartesian plane as foci, is established.

2. Let $f(x)$ be a polynomial of degree n greater than unity, and $f_p(x)$ its pth iterate. If $|x|$ is sufficiently large, $f_p(x)$ goes to infinity with p. Dr. Ritt studies the domain which is drawn to infinity. This study is made by means of a function $\varphi(x)$, introduced by L. Boettcher, which is defined by the relation

$$\varphi(x) = \lim_{p=\infty}\,[f_p(x)]^{1/n^p}.$$

It follows from the existence of this function that the domain drawn to infinity is a connected region. If none of the roots of $f'(x) = 0$ lies in this domain, the domain is simply connected. If the coefficient of x^n in $f(x)$ is unity, this simply connected domain covers a portion of the interior of the circle $|x| = 1$. If a root of $f'(x) = 0$ lies in the domain drawn to infinity, that domain is infinitely connected.

3. In a paper previously presented to the Society Professor Decker gave a proof of a theorem announced by Salmon without proof: The number of solutions of the system of equations arising from the vanishing of all determinants of the mth order that can be formed from a matrix with m rows and n columns, $m \ngtr n$, by the suppression of $n - m$ columns, the element in the ith row and jth column being of degree $\alpha_i + \alpha_j$

in $n - m + 1$ non-homogeneous variables, is K_{n-m+1}, where

$$K_l = \sum_{i=0}^{i=l} c_i \, \delta_{l-i},$$

c_l representing the sum all possible products of l different α's and δ_l the sum of all possible products of l α's, repetitions being permissible.

In the present paper the system arising from the vanishing of all determinants of the rth order that can be formed by the suppression of $n - r$ columns and $m - r$ rows, $r \ngtr m$, $r \ngtr n$, is treated. The order is shown to be

$$\begin{vmatrix} K_{n-r+1} & K_{n-r} & \cdots & K_{n-m+1} \\ K_{n-r+2} & K_{n-r+1} & \cdots & K_{n-m+2} \\ \cdots & \cdots & \cdots & \cdots \\ K_{n-2r+m+1} & K_{n-2r+m} & \cdots & K_{n-r+1} \end{vmatrix}.$$

If the degree of every element is b, the order reduces to

$$\frac{\prod_{i=0}^{i=m-r} {}_{n+i}C_{n-r+1} \cdot b^{(m-r+1)(n-r+1)}}{\prod_{j=1}^{j=m-r+1} {}_{n-r+j}C_{n-r+1}},$$

a result established by Segre.

4. Professor Glenn's paper is devoted primarily to a principle in modular invariant theory based, in a general theoretical way, upon analogy with the algebraical principle of symbolical convolution, by which, in both theories, an assigned covariant of a given degree becomes a member of a finite scale of derived concomitants of the same degree. Upon this principle the main theorems in algebraical invariant theory have been made to rest, and, in the present paper, the elements of a similar general modular theory are considered.

The methods developed are applied in deriving a complete system of formal concomitants of the binary quadratic form under the total binary group modulo 3.

The paper will be offered for publication in the *Transactions*.

5. In Professor Rowe's paper the $u_t^6 = 0$, $v_t^6 = 0$, and $w_t^6 = 0$ are three linearly independent binary sextics of the

double infinity of binary sextics to which all plane sections of the R_3^6, the rational sextic curve in ordinary space, are apolar, and $a_t^5 = 0$ is a binary quintic. The necessary and sufficient condition that the R_3^6 have a quinquesecant line is that $|\alpha u|^5 u_t = 0$, $|\alpha v|^5 v_t = 0$, and $|\alpha w|^5 w_t = 0$ hold for all values of t. This gives rise to an invariant of degree two in the four-rowed determinants of the matrix of the coefficients of the parametric equations of R_3^6. By a simple substitution this invariant may be transformed into an invariant of the rational plane sextic.

6. Professor Safford has obtained in parametric form the equations of the path of a projectile when the air resistance varies as the nth power of the velocity of the projectile. This problem was proposed by a member of the National Research Council and it was stated that the solution was known only for the case of $n = 2$, as given by de Sparre in the *Comptes Rendus,* volume 160. In that case certain approximations were made in the early stages of the discussion. In the present paper the accuracy of the results depends only upon the accuracy with which the physical constants can be determined.

7. Professor Moore defines a surface of revolution as one left invariant by a rotation and discusses the forms of such surfaces if the rotation is restricted to the type having an infinite number of invariant planes.

8. In this paper Professor Moore shows that no translation surfaces in hyperspace, except cylinders, can be expressed in more than one way as translation surfaces. The condition that the surface be developable is also discussed.

9. A necessary and sufficient condition that the space cubic

$$x_1 = at, \qquad x_2 = bt^2, \qquad x_3 = ct^3 \qquad (abc \neq 0)$$

be algebraically rectifiable is, as Dr. Curtis shows, that the cubic be a helix.

10. A (2, 1) correspondence between the points of two spaces (x) and (x') may be defined by three equations which are linear in (x'). The transformation which interchanges the two points in (x) which correspond to any point in (x') is

involutorial. By this method, Professors Sharpe and Snyder discuss various involutorial transformations, in particular the case in which the equations represent quartic surfaces through an octic curve of genus two. Each quartic surface is invariant under the transformation which is of order 31, and a similar transformation. The product of the two involutions is of infinite order, so that the surfaces are similar to but distinct from the known Fano quartic surfaces through a sextic of genus two.

11. Dr. Seely considers kernels that have an infinite number of iterated kernels of positive type with respect to all functions of the form $\alpha\varphi_i(s) + \beta\varphi_j(s)$ and $\alpha\psi_i(s) + \beta\psi_j(s)$, where the φ's and ψ's are characteristic functions, and shows that such kernels have certain properties in common with symmetric kernels.

12. Birkhoff (*Transactions,* 1908) has given a general theory of the convergence of the development of an arbitrary function $f(x)$ in series of characteristic solutions of an ordinary linear differential equation of the nth order with "regular" boundary conditions. It was found by Jackson (*Proceedings of the American Academy,* 1915) that, for an extensive class of irregular boundary conditions, hypotheses similar to those imposed by Birkhoff on $f(x)$ are very far from being sufficient to insure convergence of the corresponding expansion. Lieutenant Hopkins examines more carefully the case of the differential equation $d^3u/dx^3 + \rho^3 u = 0$, with the irregular boundary conditions $u(0) = u'(0) = u(\pi) = 0$. He finds that if a function $f(x)$ is to have a uniformly convergent development in terms of the characteristic solutions of this system, it must be analytic throughout a triangular region surrounding the origin in the complex x-plane, and must furthermore be of the particular form $x^2\varphi(x^3)$, where φ is an analytic function of its argument. That is, $f''(x)$ must be an analytic function of x, invariant under the transformation $x' = \omega x$, where ω is a complex cube root of unity. He proves conversely that if $f(x)$ does have the properties indicated, throughout a suitable region, its development will converge uniformly to the value $f(x)$ throughout any interval contained in $(0, \pi)$ and not reaching out to the terminal point $x = \pi$. He has thereby made an essential contribution to the theory of a class of series

which have some of the properties of power series on the one hand and of ordinary trigonometric series on the other, and which have further novel and striking properties of their own.

13. Dr. Kline proves the following theorem:
A necessary and sufficient condition that a closed, connected, connected in kleinem, plane point set should be a simple curve is that it divide its plane into two mutually exclusive domains.

14. The comparison of conformal geometry and equilong geometry brings out many fundamental differences as well as many analogies. In the present paper Professor Kasner defines the particular equilong transformations which play the same rôle in the equilong theory as Schwarzian reflections or conformal symmetries in the theory of functions of a complex variable. The analytic representation as well as the geometric interpretation is much simpler in the equilong theory. The group generated by symmetries is also obtained.

15. In the paper of Professor Phillips a canonical form is given for a function of two or more commutative matrices, analogous to Sylvester's formula for a function of a single matrix. This is used to represent analytic functions of two or more matrices. If z is a complex variable and Z and A matrices, it is shown that $f(Z)$ can be expanded in a Taylor's series in powers of $Z-A$, the series converging if each root of Z lies within a circle, with center at the corresponding root of A, in which $f(z)$ is analytic. A simple example is given of two commutative matrices not expressible as polynomials in the same matrix.

16. Mr. Hallett gives four pairs of independent categorical postulate sets for euclidean and double elliptic geometries of three dimensions in terms of point and order. Each pair has a large common basis. Each set is such that all the properties of a line can be deduced from it without the use of any postulates which necessitate the existence of two dimensions. In three pairs of sets a modification of order due to Kempe is used to advantage. A fifth treatment, involving fewer postulates than the others, is given for double elliptic geometry. The categoricity of the sets is proved by establishing the equivalence of sets of postulates given previously by Professor Veblen and Dr. Kline.

17. In an article in the Berlin *Sitzungsberichte*, June, 1914, Frobenius proved that if

$$x^p + y^p + z^p = 0 \tag{1}$$

is satisfied in integers prime to the prime p, then $11^{p-1} \equiv 17^{p-1} \equiv 1 \pmod{p^2}$ and if $p \equiv 2 \pmod 3$ also $7^{p-1} \equiv 13^{p-1} \equiv 19^{p-1} \equiv 1 \pmod{p^2}$. He also stated (loc. cit. page 670) that it is impossible to derive the congruence $23^{p-1} \equiv 1 \pmod{p^2}$ as a criterion for (1) by the use of his methods for the case $p \equiv 1$ (mod 3) because of the appearance of a factor $t^2 - t + 1$ in the process of eliminating certain functions t. This statement of Frobenius is incorrect. In the present paper Mr. Vandiver derives the condition $23^{p-1} \equiv 1 \pmod{p^2}$ for the solution of (1) in integers prime to p, provided p is not of the form 1 (mod 11), by using only relations given by Frobenius in the article cited above.

18. In previous papers presented to the Society under this title in January and April, 1915, Mr. Vandiver derived a certain relation involving cyclotomic integers which serves as a starting point for the derivation of a number of criteria relating to Fermat's last theorem. In the present paper the criteria $B_s \equiv 0 \pmod{p^2}$

$$s = \frac{(p-2i)\,p+1}{2} \qquad (i = 2, 3, 4, 5),$$

for the solution of equation (1) of the preceding abstract in integers prime to p are obtained, the B's being numbers of Bernoulli; $B_1 = 1/6$, $B_2 = 1/30$, etc. These criteria reduce, modulo p, to the conditions

$$B_{\frac{1}{2}(p-3)} \equiv B_{\frac{1}{2}(p-5)} \equiv B_{\frac{1}{2}(p-7)} \equiv B_{\frac{1}{2}(p-9)} \equiv 0 \pmod p$$

given by Kummer and Mirimanoff.

19. The number of classes of ideals in a cyclotomic field defined by $l^{2i\pi/p}$, where p is prime, may be expressed as the product of two integral factors, one of which (generally known as the first factor) is as follows:

$$h = \frac{f(Z)\,f(Z^3)\,\ldots\,f(Z^{p-2})}{(2p)^{\frac{1}{2}(p-3)}}$$

where

$$f(x) = r_0 + r_1 x + \dots + r_{p-2} x^{p-2},$$

$Z = e^{2i\pi/(p-1)}$, and r_i is the least positive residue of r^i, modulo p. Mr. Vandiver derives necessary and sufficient conditions that h be divisible by p^n, in terms of Bernoulli numbers, the argument used being different from those employed by Kummer and Kronecker in their treatment of the case $n = 1$.

20. In his first proof of the law of reciprocity between two ideals in a regular cyclotomic field Kummer gave the relation*

$$\left[\frac{d^{p-1} \log \omega (e^v)}{dv^{p-1}}\right]_{v=0} \equiv \frac{(\omega(1))^{p-1} - N(\omega)}{p} \pmod{p},$$

where

$$\omega(x) = a_0 + a_1 x + \dots + a_{p-2} x^{p-2},$$

$$\omega = a_0 + a_1 \alpha + \dots + a_{p-2} \alpha^{p-2},$$

$\alpha = e^{2i\pi/p}$ and $a_0, a_1, \dots, a_{p-2}$ are integers, ω being prime to the prime p. In the present paper Mr. Vandiver gives a proof of this result which differs in character from that of Kummer.

F. N. Cole,
Secretary.

NOTE ON ISOGENOUS COMPLEX FUNCTIONS OF CURVES.

BY PROFESSOR W. C. GRAUSTEIN.

(Read before the American Mathematical Society September 5, 1917.)

Let L be a continuous, closed and directed space curve, without multiple points, let $-L$ be the same curve oppositely directed, and let $F|[L]|$ be a function of L, such that $F|[-L]| = -F|[L]|$. Form the ratio

$$\frac{F|[L']| - F|[L]|}{\sigma},$$

where L' is the directed curve obtained from L by replacing an arc PP' of L by a second arc joining P with P', and σ

* *Abhandlungen, Berlin Academy,* 1859, p. 119, formula (7).