FUNCTIONS OF TWO COMPLEX VARIABLES.

Lectures Introductory to the Theory of Functions of Two Complex Variables. By A. R. FORSYTH. Cambridge University Press, 1914. xvi + 281 pp.

THE present volume consists substantially of a course of lectures delivered in the University of Calcutta in January and February, 1913, in response to a special invitation of the authorities accompanied by a stipulation that the lectures should be published. What was desired was an exposition of some subject that might suggest openings to those who had the will and the skill to pursue research. In accordance with this wish the author selected the theory of functions of two complex variables, a subject still in a preliminary stage of its development and one into the exposition of which he could incorporate a considerable body of results of his own.

No attempt is made to give a systematic discussion of the whole subject nor is attention concentrated upon one particular issue. Several distinct lines of investigation are dealt with, even though this required that their treatment should be relatively brief. The essential purpose throughout was to deal with a selection of principles and of generalities belonging to the initial stages of the theory of functions of two complex variables; and this was accompanied by the desire to establish some new results and to suggest some new problems of investigation.

The substantial results of the theory of functions of a single complex variable are assumed to be so familiar to the reader that only brief and indirect reference to them will usually suffice. Almost everywhere in the exposition the number of independent variables is restricted to two. Many of the propositions may readily be modified so as to apply to the case of n variables; but this is not always true. Consequently we have on the one hand a range of results which belong essentially to functions of more than one variable and on the other hand another range of results belonging essentially to functions of just two variables. Typical of the latter is the theory of quadruply periodic functions of two variables.

At several places the author has departed from the usual custom of dealing with only a single function of two complex variables and has considered simultaneously two such functions. Certain characteristic properties are in this way brought into fuller discussion and interesting results emerge which otherwise would not be obtained, particularly in the theory of quadruply periodic functions of two variables. He is led to the simultaneous investigation of two functions by the following considerations (pages 2–4): In the theory of functions w of a single variable z we have a relation of the form $f(w, z) = 0$ so that we may consider w as a function of z or z as a function of w, thus having a complete dual notion of inversion. But if w is a function of two complex variables z and z' we do not have such inversion in the complete sense. Consequently a second function w' is introduced. Then we have relations

$$F(w, w', z, z') = 0, \qquad G(w, w', z, z') = 0,$$

so that we may look upon w and w' as functions of z and z' or conversely upon z and z' as functions of w and w'.

This way of introducing the duality certainly leaves something still to be desired. There is nothing in the forms of the foregoing equations to suggest the pairing of w and w' on the one hand and of z and z' on the other. Moreover, if we are interested primarily in a single function of two variables we certainly have an added difficulty when we introduce another essentially unrelated function of the same two variables. It can hardly be expected that two such functions will throw essential light each upon the properties of the other. Only certain more general characteristics of a function can be expected to emerge in this way. This undoubtedly is valuable, but it also leaves something further to be desired in the way of information about the individual function primarily in consideration.

A modification of the method employed by the author seems to obviate both of these difficulties. Instead of considering only two essentially unrelated functions w and w' of the variables z and z', let us also treat in further detail the following special case, namely:

$$w = f(z, z'), \quad w' = f(z', z).$$

Under easily derived appropriate conditions which we shall not take space to state, it is clear that z and z' are inversely functions of w and w' of the form

$$z = f_1(w, w'), \quad z' = f_1(w', w).$$

We thus retain the complete duality of inversion which is so important in the case of functions of a single variable; and this is accomplished without the introduction of an essentially new function. Moreover, the pairs w, w' and z, z' play a rôle different from that of other pairs of pairs of the four quantities w, w', z, z'. All the results obtained by considering the two functions w and w' state properties belonging essentially to w alone, since w and w' are so simply related.

The first chapter deals principally with three methods of geometrical representation of two complex variables, namely: representation by points in four-dimensional space, representation by lines in ordinary three-dimensional space, representation by points in two planes. The conclusion is reached (page 14) that only two of these are even fairly useful, namely, the first and the last. To supply fully our needs in respect of geometric representation we must have some uniquely effective new idea. While awaiting its appearance we must be content with such imperfect representations as the above-named methods afford.

In the first chapter one finds also a definition of functions of two complex variables similar to the Riemann definition for the case of one variable.

In the second chapter we have a treatment of the linear transformation of complex variables:

$$w = \frac{az + bz' + c}{a''z + b''z' + c''}, \quad w' = \frac{a'z + b'z' + c'}{a''z + b''z' + c''}.$$

Such a transformation is reducible to one or another of three canonical forms according as the characteristic equation in θ, namely,

$$\begin{vmatrix} a - \theta & b & c \\ a' & b' - \theta & c' \\ a'' & b'' & c'' - \theta \end{vmatrix} = 0,$$

has three simple roots, one double root and one simple root, or one triple root. Invariant centers of the transformation are discussed. A special investigation is also given of the sort of frontiers in four-dimensional space which are left invariant as to quality by these transformations, the result corresponding to invariancy of circles under linear transformations in a single variable. If one puts $z = x + iy$, $z' = x' + iy'$, x, y, x', y' being real variables, then the invariant frontiers in question are defined by equations of the form

$$A(x^2 + y^2) + B(x'^2 + y'^2) + C(xx' + yy') + D(xy' - yx') \\ + Ex + Fy + Gx' + Hy' = K,$$

where the symbols $A, \cdots, K$ denote constants. There is also a treatment of the invariants and covariants of quadratic frontiers, and of periodic transformations.

It seems to the reviewer that a considerable number of readers will be repelled from the book by its first two chapters unless they observe the relation, or rather for the most part the lack of relation, of these two chapters to the remainder of the work. They are in large measure a reproduction of two papers by the author. Their general interest is certainly less than much of that which follows, both of the author's researches and of the matter taken from the literature. Fortunately but little of these chapters is used in building the succeeding portions of the book. Consequently it would seem well to have a list of portions suited to a first reading, similar to such a list in Forsyth's Theory of Functions of a single variable. We venture to suggest such a selection, our primary aim being to indicate to the reader how he may readily get into the meat of the subject.

In the first chapter one may take sections 1–6, 14–16. The second chapter may be omitted entirely on a first reading, or one may take sections 21–27 so as to have in mind the first properties of linear transformations. Thus after reading eleven pages from the first chapter (and possibly eleven pages from the second chapter), one is ready to enter upon the main features of the subject at the beginning of the third chapter on page 57. From this point forward one may read Chapters III, IV, V, Chapter VI up to page 169, Chapter VIII (with reference to one or two parts of Chapter VII). The remainder of the book may then be taken in any order desired. It appears to the reviewer that the reader is likely to find the book more interesting if he follows a plan of selection similar to that just indicated.

In the third chapter begins the essential development of the theory of single-valued analytic functions of two complex variables. The problem here treated is that of their expressibility in power series. Certain of the fundamental theorems for functions of a single variable are readily carried over to the case of functions of two variables; as, for instance, those connected with the Taylor and the Laurent series expansions, dominant functions, approach to every value, and certain matters concerning analytic continuation. But in the classification of singularities an essential difference emerges. There are three types of singularities; to these are given the names *pole, unessential singularity, essential singularity.* Let k, k' be a singular place for the single-valued function $f(z, z')$. If no power series $P_0(z - k, z' - k')$ in non-negative powers of $z - k$, $z' - k'$ exists such that the product

$$P_0(z - k, z' - k')\cdot f(z, z')$$

is expressible in a series of non-negative powers of $z - k$, $z' - k'$, then k, k' is called an essential singularity of $f(z, z')$. If the foregoing product can be formed so as to afford a function $P_1(z - k, z' - k')$ expansible in non-negative powers of $z - k$, $z' - k'$, then $f(z, z')$ is said to have a pole or an unessential singularity at k, k' according as it is or is not possible to choose $P_0(z - k, z' - k')$ so that $P_1(0, 0)$ shall be different from zero.

The fourth chapter is devoted principally to a consideration of the classic theorems of Weierstrass concerning the representation of a single-valued function in the vicinity of any of its various places, whether ordinary or singular. But little is said of the case of an essential singularity. For all other places the matter rests on the representation of a function in the vicinity of a zero of the function, as one sees readily from the definitions stated in the preceding paragraph. The concept of the divisibility of one function by another and the consequent theory of reducibility are also treated in this chapter. On pages 83 and 84 the author implicitly makes essential use of certain results in the theory of divisibility, whereas the theory is itself first developed on pages 112 ff.

In Chapter V one meets the following (and related) theorems analogous to corresponding ones for functions of a single variable: A function without essential singularities (either in the finite region or at infinity) is a rational function; a function which has no essential singularity in the finite region is expressible as a quotient of two functions each of which is without any singularity in the finite region.

The sixth chapter is devoted to integrals. In the first part is given a treatment of double integrals of uniform functions of two variables. Here one finds Poincaré's extension of Cauchy's main integral theorem, followed by several simple examples of a subject which awaits further development. The latter part of the chapter is concerned with integrals, both single and double, of algebraic functions, a theory to which Picard has made fundamental contributions. The author here takes the line of introducing two algebraic functions of two variables, thus following up (for the first time except for his treatment of linear transformations) his notion of the simultaneous consideration of two functions. But little more is done than to give an introduction to this problem, particularly in its preliminary formal aspects. Reference is made to Picard and Simart's Théorie des Fonctions algébriques de deux Variables indépendantes, where the case of a single function of two variables is treated in illuminating detail. The opinion is expressed that the further simultaneous consideration of two functions will lead to interesting extensions of the theory.

The seventh chapter deals with the so-called "level places" of two simultaneous single-valued functions of two variables, each function being without an essential singularity in the finite region. It is shown that two such functions simultaneously approach zero for a suitable approach of z and z' separately to suitably chosen points in their respective planes of variation, these planes being understood to contain the point infinity. Such a point is called a common zero of the two functions. If the two functions are further restricted to be independent and without common factors, then their common zeros are isolated. These results are readily applied to a consideration of the "level places" of two functions $f(z, z')$ and $g(z, z')$, namely, the common zeros of $f(z, z') - \alpha$ and $g(z, z') - \beta$, where α and β are constants.

The eighth and last chapter is devoted to the theory of single-valued periodic functions of essentially two variables. Such a function cannot have more than four linearly independent period-pairs. The cases of one period-pair and of two period-pairs are degenerate. Type forms of period-pairs for the non-degenerate cases of three and of four period-pairs are worked out in detail. The method is somewhat tedious and probably could not be extended to the case of functions of n variables; but it brings into relief the character of the various cases which initially come into consideration. The theta-functions of two variables are employed in the development of the theory. Theorems concerning algebraic relations among homo-periodic functions are derived. Simple examples of hyperelliptic functions are introduced. But the author has not attempted to expound the details of the theory of periodic functions of two variables; he has left that to the specific treatises on this subject.

A few minor misprints may be mentioned: "real" for "pure imaginary" in line 15 of page 32; "not universally" for "indeed never" in line 16 up on page 83; obvious misprints in footnote on page 75, in line 16 up on page 85, in line 15 on page 126, in line 5 up on page 236.

On the whole the book is well written and the exposition is usually clear. Some (but not many) infelicities of expression occur of which the following are examples: The author speaks several times (as on page 50) of "invariance under a single transformation only"; he employs repeatedly the redundant description "uniform, continuous and analytic" (see an instance on page 91 and the definition of "analytic" on page 59); he speaks sometimes (as on page 60) of a fixed domain as if it might be infinitesimal in extent. On page 241 and elsewhere in the following discussion it is desirable to specify *uniform* convergence as well as *absolute* convergence.

It is desirable to have an explicit definition of the region at infinity; this is not supplied by the author. It is assumed without remark (page 58 and elsewhere) that the infinite region in the z-plane and also that in the z'-plane is the infinite region of the usual complex plane; and then the infinite region for z and z' simultaneously is that in which either z or z' is infinite.

Probably minor defects of this sort (of which we do not attempt a complete list) will interfere with nothing but the comfort of the reader likely to be interested in this subject and will not seriously impair for him the value of the book.

We add a remark concerning a choice deliberately made by the author, a choice particularly unfortunate in the opinion of the reviewer.

Consider the function $e^{1/x}$. For $x = 1$ this function has the value e. When x approaches the value 1 in any way the function approaches the value e. We may say that the function possesses or acquires the value e at the point $x = 1$. Now consider what happens to the function as x approaches zero. It is easy to show that the function approaches e if x approaches zero *in a proper manner*. It is also equally easy to show that the function approaches any preassigned value if x approaches zero *in a suitable manner*. We may then say that $e^{1/x}$ *approaches* e (or any other assigned value) in case x approaches zero in a way appropriately dependent on this assigned value. Forsyth would still speak of the value e as a value which the function *acquires* or *possesses* at the point $x = 0$, using precisely the same terms as for the case of x approaching 1 and deliberately rejecting the distinction which clearly exists. See especially pages 77 and 78.

To the reviewer it seems definitely desirable to make the distinction and to mark it by a suitable choice of terminology, using *acquires* or *possesses* in the one case and *approaches* in the other. Some such distinction is generally made. There is a definite loss and no gain in refusing to make it, as most readers will probably be led to suspect by an examination of Forsyth's remarks (pages 77–79) about Picard's theorem to the effect that a non-constant entire function $f(z)$ acquires every value with at most a single exception.

Many of the results stated in the book need to be interpreted in the light of the fact that the author refuses to make the distinction in question. We illustrate by a single example. On page 198 we have the theorem: "Two independent functions, regular throughout the finite part of the field of variation, vanish simultaneously at some place or places within the whole field." Here "vanish" means to "acquire" the value zero in the sense of Forsyth's use of the term. What is actually proved is that the two functions simultaneously approach zero for an appropriate approach of z, z' to some appropriately chosen place k, k', finite or at infinity. The theorem does not look so attractive when clothed in this dress; but its true character is more readily recognized.

Similar remarks may be made about not a few other theorems. It is unfortunate for the reader to have to look through the proof of a theorem to see which sense of "acquire" gives to the theorem the best or largest meaning which the proof allows.

Finally, let us say that there can be no doubt that Forsyth has rendered his colleagues a distinct service in adding this book to his already long list of useful publications. It will be of definite value to a large number of persons interested in the theory of functions of complex variables.

R. D. CARMICHAEL.

SHORTER NOTICES.

The Geometrical Lectures of Isaac Barrow. Translated, with notes and proofs, by J. M. CHILD, B.A. (Cantab.), B.Sc. (Lond.). The Open Court Publishing Company, Chicago, 1916. xiv+218 pp.

"ISAAC BARROW was the first inventor of the Infinitesimal Calculus; Newton got the main idea of it from Barrow by personal communication; and Leibniz also was in some measure indebted to Barrow's work, obtaining confirmation of his own original ideas, and suggestions for their further development, from the copy of Barrow's book that he purchased in 1673."

"The above is the ultimate conclusion that I have arrived at, as the result of six months' close study of a single book, my first essay in historical research. By the 'Infinitesimal Calculus,' I intend 'a complete set of standard forms for both the differential and integral sections of the subject, together with rules for their combination, such as for a product, a quotient, or a power of a function; and also a recognition and demonstration of the fact that differentiation and integration are inverse operations.'"

These are the opening paragraphs of the preface to this edition of Barrow's Lectures. While the rest of the book does perhaps not justify the claims of the preface, it furnishes a very welcome addition to the generally available information concerning Barrow. It presents, in abridged form, a translation by Mr. Child of the "Lectiones Geometricæ" of 1670, of which a first English translation was published by Edmond Stone in 1735. Numerous notes, bearing upon Mr. Child's thesis, are scattered throughout the text, proofs have been added in a number of places, and there is an introduction of