

where, it is to be remembered,  $\Delta^n w = \Delta^n w_0 / 5^{n+1}$  (and  $w = w_0 / 5$ ). It should be noticed, however, that  $\delta^4 u_1$  is expressed more simply by  $.002\Delta^3 w_0$ .

The arbitrary changes just suggested give an error of only about one half of one per cent of  $\Delta^3 w_0$  in  $w_1$  (the sum of the individual values interpolated). The error in each individual value would then be much less and if only two more decimal places are used than are to be finally retained—which are all that are ordinarily necessary as found by experience—the errors would not appear at all in the results. As a check upon the work the sum of the individual values interpolated should be  $w_1$  as given originally.

It should be pointed out that the formulas derived above can not be used for “end” values; that is, if the groups of values were  $w_0, w_1, w_2, \dots, w_n$  the formulas could not be used for interpolating or breaking up  $w_0$  or  $w_n$ , for the derivation of the formulas is based upon the use of four values ( $w_0, w_1, w_2$ , and  $w_3$ ) to break up  $w_1$ ; that is, there must always be at least one group preceding the group to be broken up. To break up “end” values formula (1) could be used or the formulas for the leading term and differences to be found in the article cited.

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## NON-EUCLIDEAN GEOMETRY.

*Geometrical Researches on the Theory of Parallels.* By N. LOBACHEVSKI. Translated from the original by G. B. HALSTED. New edition. Chicago and London, Open Court, 1914. 8vo. 50 pages. Cloth, price \$1.25.

NON-EUCLIDEAN geometry had two independent discoverers: Johann Bolyai (1802–1860), a Hungarian officer in the Austrian army; and Nicolaus Lobachevsky (1793–1856), son of a Russian peasant, and graduate, professor, and rector, of the University of Kasan.\*

As early as 1823 the former had grasped the real nature of his problem, and in 1829 he sent a completed manuscript on the subject to his father, Wolfgang Bolyai, who was a

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\* The best history of non-euclidean geometry is by Bonola. It was translated into English and edited by H. S. Carslaw (Chicago, 1912).

professor of mathematics, physics and chemistry. In 1832 the father published a Latin translation of the manuscript as an appendix\* to the first volume of one of his scientific works.

It seems to have been after 1823 that the development of non-euclidean geometry occurred to Lobachevsky. This may be inferred from a passage in his book on Elementary Geometry:† “Up to the present no one has been able to find a rigorous proof of the fifth postulate of Euclid; what has been given we may only call explanations; these do not deserve to be characterized as mathematical proofs in the true sense.” The first fruit of new study in a geometry independent of Euclid’s hypothesis was a paper read before the physico-mathematical section of the University of Kasan in 1826. This paper does not seem to have been preserved, but the essential parts of it were contained in a memoir published in 1829–30. Lobachevsky indicated further applications of the new theory in three more Russian memoirs of 1835, 1835–38, 1836 (“Géométrie Imaginaire” which appeared in *Crelle’s Journal* for 1837 is really a free translation of the memoir of 1835). He endeavored to call the attention of mathematicians to his discoveries by publishing a summary of his results in a little book which appeared at Berlin in 1840 under the title: *Geometrische Untersuchungen zur Theorie der Parallellinien*.‡ It is this little work which Dr. Halsted has here given us in English form.

It is probable that Lobachevsky never even heard of Bolyai, who, in 1848, made himself thoroughly acquainted with the *Geometrische Untersuchungen*.

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\* A facsimile edition was published at Maros-Vásárhelyini in 1907. 4 + 28 pp.

† The manuscript of this work was sent to St. Petersburg in 1823 but failed of publication at that time on account of the unfavorable report of the academician N. Fuss. In 1898 the manuscript was discovered in the archives of the university of Kasan and it was published in 1909 by the Physico-Mathematical Society of Kasan (8vo. 3 + 67 pp.). Titles of sections indicate that Méray’s ideas of fusion were anticipated, many years before, by Lobachevsky: Measuring lines; Angles; Perpendiculars; Measuring solid angles; Regular polygons and polyhedrons; Equivalent triangles; Measuring right angles, triangles and other figures; Parallelograms; Measuring of pyramids and polyhedrons; Measuring the circumference and the area of a circle, the volume of the cylinder and cone; Measuring the area of the right cylinder and right cone; On the magnitude of the volume and surface of the sphere.

‡ Facsimile reprint, Berlin, 1887; [Geometrical works of Lobachevsky] (in Russian), Kasan, vol. II, 1886, pp. 553–578. French translation by J. Hoüel in *Mém. de Bordeaux*, tome 4 (1866), also separately (2d ed. 1895), and in *Recherches géométriques sur la théorie des parallèles*, Paris, 1900.

There is little doubt that Carl Friedrich Gauss (1777–1855) was the first to have a clear view of a geometry independent of the fifth postulate; but this fact “was only revealed after the works of Lobachevsky (1829–30) and J. Bolyai (1832) appeared.” To these geniuses alone, by reason of priority of publication, must belong all honor due to those confident of their results and unafraid of the world’s criticism.

In February, 1832, Wolfgang Bolyai sent a copy of his son’s work to Gauss. In acknowledging its receipt Gauss wrote, on March 6,\* somewhat as follows:

“. . . Now something concerning the work by your son. If I commence with this, ‘*that I can not praise it,*’ you may well be taken aback for a moment, but I can not do otherwise; to praise it would be to praise myself: for the whole content of the work, the path which your son has followed, and the results to which he is led, agree almost throughout with my own meditations which have occupied my mind for 30–35 years. So I am surprised to the utmost. My intention with respect to my own work, concerning which little has been put on paper up to the present, was to allow nothing to be known of it during my life time. Most people do not have any appreciation of what is involved in the subject, and I have found only few men who took up with any particular interest what I communicated to them concerning it. To be able to take such an interest it is first of all necessary to have devoted careful thought to the real nature of what is lacking, and upon this matter almost all men are far from clear. On the other hand, it was my intention some time to put the whole matter on paper, so that it should not one day perish with me.

“Greatly have I been surprised that I am spared this trouble and much do I rejoice that it is just the son of my old friend who has got the start of me in such a remarkable manner.”

And then follow a discussion of details and a presentation of his own development of some of the parts.

In a letter to Gerling about three weeks earlier Gauss characterized the younger Bolyai as a “genius of the first order.” He wrote as follows:

“. . . Noch bemerke ich, dass ich dieser Tage eine kleine Schrift aus Ungarn über die Nicht-Euklidische Geometrie erhalten habe, worin ich alle *meine eigenen Ideen und RESULTATE* wiederfinde, mit grosser Eleganz entwickelt,

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\* Gauss, Werke, Bd. 8, Göttingen, 1900, pp. 220–221.

obwohl in einer für jemand, dem die Sache fremd ist, wegen der Concentrirung etwas schwer zu folgenden Form. Der Verfasser ist ein *sehr* junger österreichischer Officier, Sohn eines Jugendfreundes von mir. . . . Ich halte diesen jungen Geometer v. BOLYAI für ein Genie erster Grösse. . . .”

Gauss was not only familiar with all the writings of Lobachevsky referred to above but he possessed all of them by 1844, and indeed, probably a year or so earlier, when he was responsible for having Lobachevsky elected as correspondent of the Royal Society of Sciences at Göttingen. Contrasting the Geometrische Untersuchungen with the memoirs, he wrote to Gerling on February 8, 1844:\* “Namentlich hat sie viel mehr Concinnität und Präcision, als die grösseren Aufsätze des Lobatschewsky, die mehr einem verworrenen Walde gleichen, durch den es, ohne alle Bäume erst einzeln kennen gelernt zu haben, schwer ist, einen Durchgang und Uebersicht zu finden.”

In a letter to his friend Schumacher in 1846 Gauss wrote further† concerning the Untersuchungen: “. . . materiell für mich Neues habe ich also im Lobatschewskyschen Werke nicht gefunden, aber die Entwicklung ist auf anderm Wege gemacht, als ich selbst eingeschlagen habe, und zwar von Lobatschewsky auf eine meisterhafte Art in ächt geometrischem Geiste. Ich glaube Sie auf das Buch aufmerksam machen zu müssen, welches Ihnen gewiss ganz exquisiten Genuss gewähren wird. . . .”

The ideas inaugurated by Lobachevsky and Bolyai did not obtain wide recognition for many years, indeed for more than a decade after Gauss's death. Shortly after Hoüel had published in 1866 the French translation of the Untersuchungen and “Appendix” together with the commendatory passages from Gauss's correspondence, the works of the almost unknown Slav and Hungarian mathematicians came into their own, and non-euclidean geometry was taken up as a serious study.

The work before us for review is described as a “new edition.” The translation seems to have been originally published,‡ with “Translator's preface,” at Austin, Texas, in *Scientiæ Baccalaureus* for February, 1891.§ A new edition

\* Gauss, Werke, Bd. 8, p. 237.

† Gauss, Werke, Bd. 8, p. 238.

‡ D. M. Y. Sommerville (Bibliography of Non-Euclidean Geometry, 1911) lists an edition published at Rolla, Mo., in 1891.

§ Vol. 1, no. 3, pp. 123-164.

with type reset, with the same preface now dated May 1, 1891, with translator's introduction and with translator's appendix was published as a Bulletin of the University of Texas.\* This same edition with date 1891 and with a "Note to the fourth edition," was apparently published in 1892. But "The Nemonic Series, Vol. IV" is also called the fourth edition though listed by Sommerville as the third. Halsted's translation was also published in the *Tōkyō Sūgaku Butsurigaku Kwai Kiji* (Proceedings of the Physico-mathematical society of Tokyo).†

Except as to binding, the edition before us is practically identical with that issued by the University of Texas. Indeed the same plates seem to have been used except for pages 5, 9 and 10. Small changes have been made on pages 7, 19, 47, 48. Pages 48 and 49 of the translator's preface have been replaced by paragraphs headed "Bibliography" and the portrait frontispiece of Lobachevsky, which has been added, seems to have been reproduced from Engel's volume.‡

It seems a pity that a little more care had not been exercised in this final revision. On page 6 we still find Henricus Billingsly instead of Henry Billingsley. While the following sentence may have been true in 1891, the date of the preface, it does not describe the conditions in 1914, the date of the volume. "Even to-day in the vast system of examination set by Cambridge, Oxford and the British government, no proof will be accepted which infringes Euclid's order, a sequence founded upon his set of axioms." Figures 13, 15, 24, 28, 31, 32, 33, still differ in some respects from Lobachevsky's originals. For example these had: lines  $EG$  and  $EH$  in Fig. 13; no arc  $C'B'$  in Fig. 15;  $LK$  as bisector of  $HC$  in Fig. 24 and so on.

The translation is faithfully and skilfully done. While the paraphrasing is quite different from the original the change has possibly contributed to clearer presentation. The long and unnecessary spacings after sections 16 and 24 do not occur in the original.

The Open Court publishing company has rendered a service

\* Austin, 1891, 50 pp.

† Vol. 5 (1892-94), pp. 6-50.

‡ Nikolaj Iwanowitsch Lobatschewskij. Zwei geometrische Abhandlungen aus dem Russischen uebersetzt, mit Anmerkungen und mit einer Biographie des Verfassers. Von F. Engel. Leipzig, 1898. Reviewed by F. S. Woods in this BULLETIN, vol. 6, 1900, pp. 339-344; 443-452.

to mathematicians in placing at their disposal English editions of Lobachevsky's and Bolyai's masterpieces.

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### FRENCH SCIENCE.

*Science and Learning in France with a Survey of Opportunities for American Students in French Universities.* An Appreciation by American Scholars. The Society for American Fellowships in French Universities, 1917. 8vo. 40 + 454 pages. (For sale by A. C. McClurg and Co., Chicago: cloth, \$1.50; stiff paper, \$1.00.)

*Exposition Universelle et Internationale de San Francisco. La Science Française.* 2 tomes. Paris, Ministère de l'Instruction Publique et des Beaux Arts, 1915. 397 + 405 pages.

*L'Histoire des Sciences et les Prétentions de la Science Allemande.* By E. PICARD. Paris, Perrin et Cie., 1916. 12mo. 49 pages. Boards. Price 60 centimes.

THE Society for American Fellowships in French Universities has announced that it is prepared to award ten or more fellowships, each of one thousand dollars annually, to American graduate students who undertake to study in French Universities. The fellowships will be given for studies in any of the following fields of knowledge: anthropology, archæology, astronomy, botany (and agriculture), chemistry, criminology, education, engineering, geography, geology, history, law, mathematics, medicine, classical philology, oriental philology, romance philology, semitic philology, English philology, philosophy, physics, political science (including economics and international law), psychology, religion, sociology, and zoology (including biology).

The awards will be based on nation-wide competition. It is planned to make the first annual award in the spring of 1918, so that the Fellow may depart for France in July, 1918, if the war has ended by that time. It has been announced that a circular of information, giving details as to the persons eligible, the form of application, and the mode of making the award, will be ready for distribution to all American institu-