AN UNDERVALUED KIRKMAN PAPER.

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THE purpose of this note is to emphasize the importance in the theory of triad systems of a Kirkman* paper, which appears to have been overlooked by all writers on this subject up to the present time.

A short explanation of the symbols employed in the paper is necessary. The symbol $Q_{x,y,z}$ denotes the greatest number of combinations that can be made with x elements, y at a time, so that no combination of z elements shall be twice employed; for brevity $Q_{x,3,2}$ is replaced by Q_x. The symbol V_x denotes the number of pairs possible with x elements that are excluded from Q_x. The symbol q_x denotes the number of triads formed with x elements, in which no duad is twice employed, q_x being not necessarily a maximum; v_x is the number of duads possible with x elements not employed in q_x. Four pairs such as 12, 23, 34, 41 forming a closed circle are denoted by the symbol C_4.

The object of Kirkman's paper is to determine the value of Q_x, and to establish the following theorems:

$$Q_x = \frac{x(x-1)}{6} - \frac{V_x}{3},$$

where

$$V_x = \frac{x}{2} + 3k + 1 \text{ if } x = 6n - 2;$$

$$V_x = 6k + 4 \quad \text{`` } x = 6n - 1;$$

$$V_x = \frac{x}{2} \quad \text{`` } x = 6n, 6n + 2;$$

$$V_x = 0 \quad \text{`` } x = 6n + 1, 6n + 3;$$

$$2^m(2k + 1) = n; \; n, m, x, k \text{ are integers } \geqq 0.$$

The case of special interest to us is that in which $V_x = 0$,

* *Cambridge and Dublin Math. Journal*, vol. 2 (1847).

for only under this condition does a triad system on x elements exist. Kirkman proves the two following propositions:

I. If $x = 2n + 1$ and $V_x = 0$, then $V_{2x+1} = 0$, and $v_{2x-1} = C_{2x-2}$; that is, if x is odd and a triad system on x elements exists, then a triad system on $2x + 1$ elements may be constructed which contains a circle of $2x - 2$ pairs chosen from $2x - 1$ elements. This theorem gives the construction for one type of the headed triad systems Δ_7, Δ_{15}, Δ_{19}, $\Delta_{27}, \cdots$.

II. If $x = 2n$ and $v_{x+1} = C_x$, then $V_{2x+1} = 0$, and

$$v_{2x-1} = C_{2x-2};$$

that is, if x is even and if from $x + 1$ elements a partial triad system can be formed such that the unemployed pairs form a closed circle of x pairs, then a triad system on $2x + 1$ elements may be constructed which contains a circle of $2x - 2$ pairs chosen from $2x - 1$ elements. This theorem gives the construction for one type of triad systems Δ_9, Δ_{13}, $\Delta_{25}, \cdots$.

These two propositions enable Kirkman to prove that a triad system exists for every value of x of the form $6n + 1$ or $6n + 3$. This result has, in general, been attributed to Reiss,* but, as the Reiss paper was not published until 1859, twelve years after that of Kirkman, the credit properly belongs to Kirkman. A comparison of the Reiss paper with that of Kirkman reveals a remarkable identity of method in the work of these two men. The construction given by Reiss for a triad system on $2x + 1$ elements, x being an odd number, is a perfect duplication of the work of Kirkman. The Reiss construction for a triad system on $2x + 1$ elements, x being an even number, may seem at first sight to differ from that given by Kirkman, but investigation reveals the fact that the two methods are essentially the same and always furnish congruent triad systems. This curious duplication by Reiss of the Kirkman work gives rise to the question as to whether Reiss had a knowledge of this Kirkman paper, or whether the problem was of such a type that two investigators would inevitably have followed the same line of argument.

In Crelle's *Journal für Mathematik*, vol. 45 (1853), Steiner proposes some problems in combinations now generally known as the Steiner combination problems. The first of these problems is only a special case of the Kirkman symbol $Q_{x,y,z}$,

*M. Reiss, "Ueber eine Steinersche kombinatorische Aufgabe." *Crelle's Journal*, vol. 56 (1859), p. 326.

and the remaining problems are slight modifications of the Kirkman problem. Since Kirkman published the general problem ($Q_{x,y,z}$) as early as 1844 in the *Ladies' and Gentlemen's Diary*, and, by 1850, had already contributed two papers on the subject to the *Cambridge and Dublin Mathematical Journal*, the *Kirkman* combination problem would seem to be a more appropriate name for the so-called Steiner problem (*a*). Moreover, the Steiner problems (*b*), (*c*), (*d*), (*e*), should be designated as the Steiner modifications of the Kirkman combination problem.

Netto makes no mention of this Kirkman paper either in his Vorlesungen or in his article on Tripelsysteme in the Encyklopädie der Mathematischen Wissenschaften. In the Encyklopädie article, Netto refers only to two papers by Kirkman, one, in volume 7 (1852), and the other, in volume 8 (1853) of the *Cambridge and Dublin Mathematical Journal.* Now, volume 7 contains no paper by Kirkman on any subject, and the paper in volume 8 is Kirkman's third paper on the problem $Q_{x,y,z}$. That Netto in preparing an article for the Encyklopädie should have overlooked entirely Kirkman's two earlier papers, published in the *Journal* in volumes 2 and 5 respectively, seems the more astounding, since he had at his disposal, not only a very good library at Giessen, but also all of the great mathematical libraries of Germany. The comparative rarity of the *Cambridge and Dublin Journal* is the apparent excuse for the ignorance of Kirkman's work and the exaltation of Steiner and Reiss, but this is a weak argument for Netto when he undertakes an Encyklopädie article.

Another point worthy of attention in the Kirkman paper is represented in the notation $v_{2x-1} = C_{2x-2}$. Interpreted in terms of the special case of fifteen elements, Kirkman shows that a triad system constructed by his method contains a closed circle of twelve pairs involving thirteen elements. Here, then, is the first example of what Professor F. N. Cole, in his work on triad systems, has designated as the dodekad interlacing of the triads. Investigation shows that this Kirkman system contains 56 of these dodekad interlacings, and that among the 80 non-congruent triad systems on fifteen elements enumerated by Professor Cole, 78 contain interlacings of this type.

The remaining propositions in this Kirkman paper concern the determination of Q_x when a triad system on x elements does not exist, and, therefore, are steps towards the solution of the hitherto so-called Steiner problem (a).

The school-girl problem is merely an example which originated in the development of this paper on combinations, and Kirkman justly complained of the almost total eclipse of this paper in the wide popular interest aroused by the school-girl problem. The eclipse appears to have continued up to the present day, since no mention is made of this Kirkman paper by Steiner, Reiss, Netto, or by any of the recent writers on triad systems.

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EVERY one knows that one of the noted proofs of the impossibility of an algebraic solution of the general quintic equation is due to Wantzel. Nevertheless histories of mathematics and biographical dictionaries are silent regarding his life. The eleven papers listed in Poggendorff's Handwörterbuch as due to "Pierre Laurent Wantzell" do not include the proof in question, and a query is raised in a footnote regarding another "Wantzell"; but nowhere does Poggendorff refer to a "Wantzel." Text-books on algebra and the theory of equations do not give Wantzel's full name. The reader is thus left without positive information as to the author of "Wantzel's proof." His name suggests German nationality, as does the name of "Mannheim," of slide-rule fame. Yet both these men were born in Paris and passed their lives at the Polytechnic School there.* Born in 1814, Wantzel died prematurely in 1848. He is the "Pierre Laurent Wantzell" of Poggendorff but in his published articles his name is always spelled

* On the life of Wantzel, see Barré de Saint-Venant in *Nouvelles Annales de Mathématiques* (Terquem et Gerono), vol. 7 (1848), pp. 321–331; A. de Lapparent in Ecole polytechnique, Livre du Centenaire, 1794–1894, vol. I., Paris, 1895, pp. 133–135, see also pp. 63–65, 190; Gaston Pinet's Ecrivains et Penseurs Polytechniciens, 2e éd., Paris, 1902, p. 20; Charles Sturm in *Comptes rendus hebdomadaires des Séances de l'Académie des Sciences*, Paris, vol. 28 (1849), pp. 66, 67.