THE ELEVENTH REGULAR MEETING OF THE SOUTHWESTERN SECTION.

THE eleventh regular meeting of the Southwestern Section of the Society was held on December 1, at Norman, Oklahoma, in the administration building of the University of Oklahoma, Professor S. W. Reaves presiding. The sessions opened at 10.30 A.M. and 2.30 P.M. respectively. Attending members were entertained at a smoker on the previous evening and at lunch on the day of the meeting. There were present fourteen persons, including the following eleven members of the Society:

Professor Nathan Altshiller, Professor C. H. Ashton, Professor Henry Blumberg, Professor E. W. Davis, Professor E. P. R. Duval, Professor E. R. Hedrick, Professor O. D. Kellogg, Mr. B. B. Libby, Mr. E. D. Meacham, Professor S. W. Reaves, and Professor W. H. Roever.

It was decided that the next meeting of the Section should be held on November 30, 1918, at Columbia, Missouri, and that the program committee should be Professors E. R. Hedrick (chairman), W. C. Brenke, O. D. Kellogg (secretary).

The following papers were presented at this meeting:

(1) Professor A. M. HARDING: "Rational plane anharmonic cubics."

(2) Professor NATHAN ALTSHILLER: "On the I-centers of a triangle."

(3) Professor R. D. CARMICHAEL: "Fermat numbers $F_n = 2^{2^n} + 1$."

(4) Dr. P. R. RIDER: "An intrinsic equation solution of a problem of Euler."

(5) Professor O. D. KELLOGG: "Interpolation properties of solutions of certain differential equations."

(6) Professor S. LEFSCHETZ: "On multiple integrals belonging to an algebraic variety."

(7) Professor NATHAN ALTSHILLER: "On the Teixeira construction of the unicursal cubic."

(8) Professor HENRY BLUMBERG: "Non-measurable functions connected with functional equations."

(9) Professor HENRY BLUMBERG: "A theorem on semi-continuous functions."

(10) Professor J. E. MCATEE: "Polynomial modular invariants of a binary quadratic."

(11) Professors E. R. HEDRICK and LOUIS INGOLD: "A generalization of Bessel's inequality and related formulas."

(12) Professor W. H. ROEVER: "Geometric explanation of a certain optical phenomenon."

Professor McAtee's paper was communicated to the Society through Professor Fleet. In the absence of the authors the papers of Professors Harding, Carmichael, Lefschetz, and McAtee were read by title, and Dr. Rider's paper was presented by Professor Roever.

Abstracts of the papers follow in the order of their titles above.

1. In a recent paper (*Giornale de Matematiche,* volume 54, 1916) Professor Harding has studied certain projective properties of anharmonic curves and has found the coordinates of the invariant triangle in terms of the invariants of a certain differential equation. The present paper is limited to a discussion of the properties of rational anharmonic cubic curves. It is shown that each of these curves has one cusp and one point of inflection, and that the invariant triangle is formed by the cuspidal tangent, the inflectional tangent, and the line joining the cusp to the point of inflection.

2. Professor Altshiller's paper will soon appear in the *American Mathematical Monthly.*

3. In this paper, Professor Carmichael gives elementary proofs of all the essential known facts about the Fermat numbers $F_n = 2^{2^n} + 1$ and a derivation of some new results of minor importance.

4. The following problem was proposed and solved by Euler: Given two points P_0 and P_1, and directed lines P_0Q and QP_1 through them, to determine an arc tangent to these two lines at P_0 and P_1, which with its evolute and its normals at P_0 and P_1 will enclose the minimum area. Making use of a method developed by Radon for minimizing the integral $\int_{s_0}^{s_1} F(x, y, \theta, \kappa)ds$, Dr. Rider gives an intrinsic equation solution of the problem which is much shorter than the solution ordinarily given.

5. Continuing the work reported on at the summer meeting, (see BULLETIN, November, 1917, page 62) Professor Kellogg

extends to the case of the general self-adjoint homogeneous boundary conditions the results of Liouville on the possession of the interpolation property (D) by the solutions of linear differential equations of second order. The results will appear in the *American Journal of Mathematics*.

6. Let $F(x, y, z, t) = 0$ be the equation of an algebraic three-dimensional variety in S_4 and consider the integrals

$$J = \int\int\int \frac{P(x, y, z, t)}{F_t'} dxdydz, \quad J = \int\int \frac{P(x, y, z, t)}{F_t'} dxdy$$

belonging to F and to its hyperplane sections $z = C$, or H_z. The fundamental theorem proved by Professor Lefschetz in his paper is this: The periods of J_z are algebraic functions of z. This makes it possible to replace J as far as its periods are concerned by a certain abelian integral. The number of these periods is found to be $R_3' = I + 2R_2 - 3R_1 - 4$ (I invariant of Zeuthen-Segre, R_i i-dimensional connectivity). By comparing with a result of Alexander it is seen that $R_3' = R_3 - R_1$, which agrees with the existence of R_1 tridimensional cycles relative to which the periods of J are zero. The definition of triple integrals of the second kind is about the same as Picard's for algebraic surfaces. Their reduction and enumeration is closely related to a class of double integrals of the type

$$\int\int Udydz + Vdzdy + Wdxdy; \qquad \frac{\partial U}{\partial x} + \frac{\partial V}{\partial y} + \frac{\partial W}{\partial z} = 0,$$

and to an invariant number λ analogous to Picard's number ρ. The number of integrals of the second kind turns out to be $\rho_0 = R_3' - \lambda$. Extensions to higher varieties are also considered. A summary of these results appeared in the *Comptes Rendus* of May 29, 1917.

7. Consider a point O, a line s, a conic C, and two points D and A on s and C respectively. A variable line through A meets s in P and C again in B. Let $M \equiv (OP, DB)$. The locus of the point M is, in general, a unicursal cubic having O for its double point, passing through D and through the points common to s and C. Conversely, with an arbitrarily chosen straight line an infinite number of conics may be associated in

order to generate a given unicursal cubic by the above method, provided the straight line does not pass through the double point of the cubic and is not an inflectional tangent. Professor Altshiller proves synthetically and discusses this proposition, which is a generalization of a theorem due to F. Gomes Teixeira (*Nouvelles Annales de Mathématiques,* August, 1917, pages 281–284).

8. At the 1916 meeting of the Southwestern Section, Professor Blumberg communicated—among other things—the fact that discontinuous solutions of the functional equation $f(x+y) = f(x) + f(y)$ are necessarily non-measurable. This holds essentially also for the equation $f(xy) = f(x)f(y)$. These examples suggest the generalized relation

$$f[\varphi(x, y)] = \varphi[f(x), f(y)],$$

where φ is to be regarded as a fixed function of two variables and f as the variable solution of the functional equation. Under suitable assumptions for φ, a large class of functional equations is obtained whose discontinuous solutions are necessarily non-measurable.

9. The theorem of Professor Blumberg's second paper is as follows: Let φ_{ab} be a monotone decreasing, real interval-function; i. e., φ_{ab} is a real number for every (closed) interval (a, b), and, in addition, $\varphi_{ab} \leqq \varphi_{cd}$ for (a, b) within (c, d). Moreover, let φ_{ai} have a finite lower bound. Let $\varphi(x)$ be the point function associated with φ_{ab}; i. e., $\varphi(x)$ is the greatest lower bound of all φ_{ab} such that x is in the interior of (a, b); let $\varphi^{(r)}(x)$ —the symbol (r) designating "right"—be the greatest lower bound of all φ_{xb}, and $\varphi^{(l)}(x)$, the greatest lower bound of all φ_{ax}. Then each of these three functions of x is an upper semicontinuous function—this part of the theorem is in essence not new—and $\varphi(x) = \varphi^{(r)}(x) = \varphi^{(l)}(x)$ except at most in a countable set. It follows that the saltus function $s(x)$ is, except at most in a countable set, indentical with $s^{(r)}(x)$ and $s^{(l)}(x)$— whose meaning is self-explanatory—and similarly for the f-saltus, d-saltus, etc. (Cf. author's paper, "Certain general properties of functions," *Annals of Mathematics*, March, 1917.) As a very special case, it follows that a function continuous on the right (left) except at most at the points of a countable set is everywhere continuous except possibly at the points of a countable set.

10. Professor Dickson has found certain polynomial modular invariants and properties of such invariants for the case of a binary n-ic and a prime modulus P. In this paper Professor McAtee generalizes these invariants for the case of a binary quadratic and modulus a power of a prime, P^λ. Then these invariants are specialized for the case $P = \lambda = 2$ and a fundamental system is exhibited modulo 4.

11. With a view toward applications to the expansion of functions in terms of given functions, Professors Hedrick and Ingold have been led to study the properties of a general linear distributive operation $L[f(x), \varphi(x)]$ on a pair of functions $f(x)$ and $\varphi(x)$. In this paper, it is shown that such an operation leads at once to a general formula of which Bessel's inequality is a special case. Other related formulas are also generalized.

A similar operator, defined only for the product $f(x) \cdot \varphi(x)$ has been studied by Moore. (See BULLETIN, volume 18, pages 334–362.)

12. Professor Roever gives a geometric explanation of elliptical light curves seen in a highly scratched plate illuminated by a point source.

O. D. KELLOGG,
Secretary of the Sectiou.

NOTE ON CONJUGATE NETS WITH EQUAL POINT INVARIANTS.

BY DR. G. M. GREEN.

(Read before the American Mathematical Society, September 4, 1917.)

IN my second memoir on conjugate nets on a curved surface,* I gave a new characterization of conjugate nets with, equal Laplace-Darboux invariants. The theorem as there stated, however, is not quite complete, and it is the purpose of this note to supply the necessary refinement, as well as to generalize the theorem and put it into relation with another,

* *Amer. Journal of Mathematics*, vol. 38 (1916), pp. 287–324. See in particular p. 313.