tions of partial differential equations and Green's functions. A brief but comprehensive résumé of recent investigations concerned with these two topics is given. The writer points out that the important difference between the Green's function and the elementary solution of the corresponding differential equation, corresponds to a similar difference between Cauchy's problem and Dirichlet's problem. That is to say the Green's function, like Dirichlet's problem, depends very closely on the form of a certain surface or hypersurface, whereas the elementary solution, like Cauchy's problem, does not. From this fact it is readily seen that considerations of analysis situs will play an important rôle in the study of Green's functions. Hence these functions are related to all the principal topics of the preceding lectures, and therefore, as the writer expresses it, a discussion of them forms an appropriate conclusion.

CHARLES N. MOORE.

Die elliptischen Funktionen und ihre Anwendungen. Erster Teil. By ROBERT FRICKE. Leipzig, B. G. Teubner, 1915. x+500 pp. Price 22 Marks.

THE present volume is the first of a series of three which Dr. Fricke proposes to write on the elliptic functions and their applications. It appeared in October, 1915, and is devoted to the function theoretic and analytic bases of the theory of elliptic functions. One would naturally expect that a treatise on elliptic functions from the pen of Dr. Fricke would follow the lines of thought developed by Klein and his students thirty-odd years ago. Consequently, on turning the pages of the present volume, one is not surprised to be reminded again and again of modes of thought, of formulas, and of geometric diagrams made familiar through the Klein-Fricke Modulfunktionen. Dr. Fricke refers to this when he writes in the preface: "That I should adhere in the main to the methods of presentation, the use of the invariant theory, geometric representation, and so forth, which more than thirty years of close scientific companionship with my teacher and friend F. Klein have made my own, I may regard as self-evident."

The introduction, consisting of 105 pages, is devoted to an exposition of theorems concerning analytic functions of a single complex variable. This material is made to lead up to a statement of the basic problems of the theory of elliptic functions; namely, a study of the algebraic functions and their integrals on a Riemann surface whose deficiency is $p = 1$, and a classification of these functions into essentially different stages (Stufentheorie). The introduction closes with an application of the Fuchs' theory to linear differential equations of the second order, in particular to the hypergeometric differential equation and hypergeometric series.

The remainder of the volume is divided into two parts (Abschnitte), of which the first, consisting of five chapters, is devoted to the theory of elliptic functions of the first stage (Stufe). This, as we know, is the "Weierstrass theory" with its characteristic rational invariants g_2, g_3 and the corresponding normal form of the integral of first kind

$$u = \int \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}.$$

The discussion contains the usual materials with the invariant theory and the group theory always in the foregound. The first part closes with a discussion of the "differential equation of the periods" and the related Schwarz differential expression.

The second part, consisting of three chapters, opens with a discussion of irrational invariants of the algebraic form $f(z)$ of the fourth degree and we soon reach the characteristic irrational invariant λ (anharmonic ratio) of the second stage, and the irrational invariant μ (octahedral irrationality), characteristic of the fourth stage, together with the corresponding normal forms of the integral of first kind

$$u = \int \frac{dz}{\sqrt{z(1 - z)(1 - \lambda z)}},$$

$$u = \int \frac{dz}{\sqrt{(1 - z^2)(1 - \mu^4 z^2)}}.$$

The first of these is the Riemann normal integral, and the second resembles very closely the Legendre-Jacobi normal form, but our author prefers to call it the Abel normal form, since Abel used it in his Recherches sur les Fonctions elliptiques. The Legendre-Jacobi normal form differs from the Abel normal form in having k^2 in place of μ^4. Moreover the Jacobi elliptic functions belong to the second stage in the Stufentheorie and not to the fourth.

The explicit statement of Klein's Stufentheorie occurs in the second chapter. This theory depends upon the group of linear substitutions $\Gamma^{(u,\,\omega)}$, that is,

$$(1)\qquad \begin{aligned} u' &= u + m_1\omega_1 + m_2\omega_2, \\ \omega_1' &= \qquad \alpha\omega_1 + \beta\omega_2, \\ \omega_2' &= \qquad \gamma\omega_1 + \delta\omega_2, \end{aligned}$$

where m_1, m_2, α, β, γ, δ are integers and $\alpha\delta - \beta\gamma = 1$, and its congruence subgroups; the principal congruence subgroup of nth stage being defined by the congruences

$$(2)\quad m_1 \equiv 0,\; m_2 \equiv 0,\; \alpha \equiv \delta \equiv 1,\; \beta \equiv \gamma \equiv 0 \pmod{n}.$$

The group $\Gamma^{(u)}$; that is,

$$u' = u + m_1\omega_1 + m_2\omega_2,$$

and the modular group $\Gamma^{(\omega)}$ are considered separately. The principal congruence subgroup of nth stage of $\Gamma^{(u)}$ is defined by the first two congruences of (2), and the principal congruence subgroup of nth stage of $\Gamma^{(\omega)}$ is defined by the last four congruences of (2). The theory of the modular group and its principal congruence subgroups is contained in the Modulfunktionen (1890). The definition of the discontinuity domains for the principal congruence subgroups of nth stage of $\Gamma^{(u)}$ and of $\Gamma^{(\omega)}$ follow, and we are led, finally, to the following definition of an elliptic function of nth stage.

An elliptic function of nth stage is a uniform homogeneous function $\psi(u/\omega_1, \omega_2)$ of integral dimensions d of the three arguments u, ω_1, ω_2 which is invariant (covariant) under the substitutions of the principal congruence subgroup of nth stage of $\Gamma^{(u,\,\omega)}$; for fixed values of ω_1, ω_2, it is an analytic function of u, free from essential singularities in the discontinuity domain of the principal congruence subgroup of nth stage of $\Gamma^{(u)}$ and is defined for the entire u-plane except for $u = \infty$; and for a fixed value of u/ω_2

$$\omega_2^{-d}\psi(u/\omega_1, \omega_2) = \psi\left(\frac{u}{\omega_2}\middle|\,\omega, 1\right) \quad \left[\omega = \frac{\omega_1}{\omega_2}\right]$$

is an analytic function of ω, free from essential singularities within the discontinuity domain of the principal congruence subgroup of nth stage of the modular group $\Gamma^{(\omega)}$, and is defined for the positive ω-half plane.

Nearly a quarter of a century ago Professor Klein expressed the hope that he might sometime be able to treat the whole theory of elliptic functions from this point of view,* and the present text, when completed, may be regarded as in a sense a fulfillment of that hope.

The Jacobi functions, sn w, cn w, dn w, now appear as elliptic functions of the second stage and the remainder of the volume is devoted to their properties, their relations to the Weierstrass functions, and to the properties of the allied theta functions.

Dr. Fricke's style is well known the world over and needs no comment. The text is free from errors, due credit being given by the author to Fräulein Dr. H. Petersen for careful proof-reading. The book appeared from the Teubner press "in spite of all the difficulties of the time, without unusual interruptions." In conclusion we may express the hope that nothing will interfere with the publication of the two remaining volumes.

L. WAYLAND DOWLING.

Problems in the Calculus, with Formulas and Suggestions. By DAVID D. LEIB, Ph.D. Boston, Ginn and Company, 1915. xii+224 pp. Price $1.00.

THE first impression conveyed by the title of this book, that it may be a collection of problems of the sort discussed by Professor Archibald in the BULLETIN of June, 1914, is corrected by the first few words of the preface. The book is said to be "the outgrowth of lists of problems prepared by the author to supplement the textbook," and in it we find "a supplementary list of workable problems on any topic ordinarily included in a general course in the calculus."

Preceding each set of exercises is a brief statement of the formulas and methods applicable thereto, with warnings against some common errors. In fact it seems to the reviewer that too careful a classification and too definite directions are given to permit, much less encourage, the development of initiative, versatility, and flexibility on the part of the student. When he has worked through a reasonable number of examples in his text-book, where properly enough most of them are carefully classified, the best additional preparation for the

* Evanston Colloquium (Aug. 28–Sept. 9, 1893). Lecture X.