volume closes with 87 pages on functions of a complex variable, including the fundamental theorem of algebra and conformal representation.

The third volume, on the integral calculus, includes such matters as Cauchy's theorem, the theorem of Gauss on transformation of a volume integral into a surface integral, Green's and Stokes's theorems, with a short discussion of vectors in connection with the latter. There is an admirable chapter on improper integrals, that is, on integrals of discontinuous functions and integrals with infinite limits. The final chapter, on differential equations, is confined to those equations which commonly occur in the applications, but contains a good treatment of the geometric interpretation of a differential equation and the existence proof.

The volumes contain very much more material than could possibly be included in any ordinary university course even in Germany, but at the same time there is scarcely anything which should not be of essential value to every student of physics or of engineering. As a supplementary hand-book, to which the teacher could refer for a sound and clear discussion of fundamental principles, it is all that could be desired, and is quite the best book of this kind, so far at least as the students referred to are concerned, which has come to the reviewer's attention.

M. W. HASKELL.

Functions of a Complex Variable. By E. J. TOWNSEND. New York, Henry Holt and Company, 1915. vii + 384 pp. 8vo. Price $4.00.

THERE has been a noticeable dearth of text-books in the English language on the elements of the theory of functions of a complex variable. The student with an easy command of German and French has found a rich and delightful literature, while his companion still in the period of language difficulties has had little opportunity to choose his reading in function theory in accordance with his individual tastes and requirements. But within the last three years there have appeared English translations of the classic works of Burkhardt and Goursat, a very full volume by Pierpoint, and the book we have before us for review.

The most obvious advantage of Townsend's treatment seems to be the absence of the synoptic character common to texts written for advanced students of mathematics. Apparently no knowledge of matters beyond the elementary calculus is assumed, unless the existence of the definite integral $\int^b f(x)dx$, where $f(x)$ is a continuous function of the real variable x in the interval $a \leqq x \leqq b$, may be regarded as such an assumption. The general theory of the line integral based on the foregoing existence theorem is completely developed. In this connection it may be remarked that the author's use throughout the book of an *ordinary* curve (precedent is cited for the terminology, page 47) as the path of integration makes it practicable to attain complete rigor in the proofs with a minimum of complexity. The ordinary curve has as its most convenient property the quality of being monotone by segments finite in number, a property not possessed by the more general *regular* curve used in Osgood's treatise. To the beginner the fundamental theorems of function theory are fully as satisfactory stated in terms of the simpler curve. The zest for generalization naturally comes at a later period in his study. We have here then a treatment of the function theory which appears to go to the bottom of things without being led into delicate generalizations and which requires on the part of the reader only a thorough first course in calculus and a certain maturity of mind.

The book is unique in its extraordinarily detailed treatment of the elementary functions (55 pages). It thus affords opportunity for an intimate acquaintance with old friends from a new point of view and offers an interesting concrete field of application for many of the general theorems. This is all the more important in view of the fact that "the material chosen deals for the most part with the general properties of functions of a complex variable, and but little is said concerning the properties of some of the more special classes of functions, as for example elliptic functions, etc." It may well be questioned whether a first course in function theory should include among its objects the introduction of a variety of new functions or whether the emphasis should be rather on a more profound acquaintance with the old functions and a clearing up of difficulties naturally arising in the study of the calculus, incidentally laying foundations for work in the field of elliptic functions, etc., if the occasion for such work presents itself. The latter seems to be distinctly the view-point of the author of the present volume.

Chapter I is devoted to the arithmetic of complex numbers including the geometric representation of the processes of addition, subtraction, multiplication, and division. Chapter II is concerned with the notion of a limit in the complex domain. A number of theorems useful later on are established. For example, it is proved that if $f(z)$ is continuous in a closed region, this continuity is uniform—a theorem not always presented in books of an elementary character. It is in this chapter only that one finds the author yielding to the temptation to generalize beyond the immediate needs of the student. The definition given of continuity in a boundary point of a region is such that the continuity of a function in a closed region does not imply by definition the continuity of the set of boundary values itself. Instead of merely supplying the proof that the set of boundary values is actually continuous, the author states and proves the more general theorem (page 38) that "if $f(z)$ is defined for a closed region S and converges uniformly along an arc C of the boundary of S, then $f(t)$ is continuous, where t denotes the values of z on C." So far as later applications are concerned this theorem seems to be needless. However, the whole chapter is a vigorous exercise in ϵ, δ reasoning and the added refinement may increase its disciplinary value.

Chapter III, a long chapter devoted to the fundamental integral theorems, is one of the most interesting in the book. The existence of a definite integral of a continuous function $f(z)$ over an ordinary curve C having been once established by means of the relation $\int_C f(z)dz = \int_C udx - vdy + i\int_C vdx + udy$, $f(z)$ is thereafter kept intact. The Cauchy-Riemann differential equations are not given until near the end of the chapter and are then derived by means of the very integral theorems which usually depend on them. The development of the integral theorems starts with the Goursat proof of the Cauchy theorem, or as the author calls it, the Cauchy-Goursat theorem, a title which seems proper since Goursat really contributed a new theorem rather than a new proof of an old theorem. Then follows the derivation of the Cauchy integral formula and the proof that the mere existence of a derivative of a function of a complex variable in a region implies the continuity of the derivative.

Morera's converse of the Cauchy theorem is given particular prominence and finds wide application. Using this theorem it is easily shown that the necessary and sufficient condition for a continuous function $f(z) = u + iv$ to be holomorphic is either that $f(z)$ satisfy the integral equation $\int_C f(z)dz = 0$ or that u and v have continuous first partial derivatives satisfying the Cauchy-Riemann differential equations. The integral theorem renders unnecessary the usual ϵ, δ proof of the sufficiency of the differential equations. The chapter closes with the usual theorems concerning Laplace's differential equation and a brief discussion of its significance in mathematical physics.

In Chapter IV the elementary functions are mapped in greater detail than in any other English text with which the reviewer is familiar. The definition of the exponential function as

$$e^z = \lim_{n=\infty} \left(1 + \frac{z}{n}\right)^n$$

and the definitions of the sine and cosine functions as

$$\sin z = \frac{e^{iz} - e^{-iz}}{2i}, \qquad \cos z = \frac{e^{iz} + e^{-iz}}{2},$$

permit the discussion of the properties of these functions in advance of the introduction of power series. In addition to the usual elementary functions several particular cases of the function

$$w = \log \frac{(z - \alpha_1)^{k_1}(z - \alpha_2)^{k_2} \cdots (z - \alpha_n)^{k_n}}{(z - \beta_1)^{\lambda_1}(z - \beta_2)^{\lambda_2} \cdots (z - \beta_n)^{\lambda_n}}$$

having important physical applications are mapped and the physical interpretations pointed out.

Chapter V, on linear fractional transformations, is sufficiently full to serve as a foundation for the theory of automorphic functions. The figures in illustration of the process of stereographic projection are unusually clear and accurate and should be a great aid at this point to the student's imagination.

Chapter VI, on infinite series, can be criticized only on the ground that it comes so late (approximately at the beginning of the second half of the volume). The author's purpose seems to have been to do first everything that can be conveniently done without the use of power series and then, having once introduced power series, to make rapid and continual use of them in developing the general analytic function theory. Logically this method is beyond criticism and it may have pedagogical advantages. In this chapter the usual theorems on power series are developed, closing with Taylor's expansion of a function holomorphic in a given region.

Chapter VII takes up the theorem on Taylor's expansion and out of it develops the notion of analytic continuation and the general definition of an analytic function. This discussion is as clear cut and accurate as one often finds. Particularly to be noted, as serving to dispel any hazy notions the student may have at this stage, is the comparison by means of examples of the properties of functions of a real variable which are differentiable infinitely many times with the properties of analytic functions of a complex variable—also differentiable infinitely many times. For example, the function

$$f(x) = e^{-1/x^2},$$

considered zero when x is zero, has infinitely many derivatives in the real domain for all real values of x. Nevertheless

$$\lim_{x=0} \frac{f(x)}{x^k} = 0,$$

where k is any integer however large. This it is pointed out is in sharp contrast to the behavior of a single valued analytic function at a zero point, in that the zero points of the latter are necessarily of definite positive integral order. Several examples, such as the series

$$E(z) = \frac{z}{1-z^2} + \frac{z^2}{1-z^4} + \frac{z^4}{1-z^8} + \cdots,$$

convergent except for values of z upon the unit circle, are given to make clear the distinction between an analytic function and a function defined by an analytic expression.

The notion of analytic function having been established, zero points, poles, and essential singular points are defined and the usual theorems are included. Regular and singular points at infinity, Laurent's expansion, and the theory of residues are next in order. We then have a full discussion of the properties of rational functions, with a proof of the fundamental theorem of algebra. Passing to transcendental functions, the Mittag-Leffler theorem in its simplest form is proved. An innovation in connection with this proof consists in the introduction of a figure as an aid to the reader in following the analytic reasoning. The function $\zeta(z)$ and the elliptic function $\wp(z) = -\zeta'(z)$ are used in illustration of the theorem. Such properties of infinite products as are essential to an understanding of Weierstrass's primary factors are developed. The chapter closes with a brief discussion of the properties of simply and doubly periodic functions.

Chapter VIII, the last, is devoted to a brief but fairly comprehensive treatment of the properties of multiple-valued functions.

On the whole the book is well coordinated with our undergraduate courses and covers just about the ground in function theory which the first year graduate student of mathematics should get well in hand.

H. B. Phillips.

Plane and Spherical Trigonometry and Tables. By G. Wentworth and D. E. Smith. Boston, Ginn and Company, 1914. 230 + 104 pp.

Quoting from the preface, this is "a work to replace the Wentworth Trigonometry which has dominated the teaching of the subject in America for a whole generation." . . . "With respect to sequence the rule has been followed that the practical use of every new feature should be clearly set forth before the abstract theory is developed."

In several particulars the book could be made more useful for students intending to pursue mathematics further. For instance, no mention has been made of Argand's diagram or of hyperbolic functions, though the logarithms of negative numbers are unusually well treated. This excellence is balanced by the unfortunate use of negative characteristics which will lead the student into endless trouble later.

Inverse function theory merits more extensive treatment even at the cost of fewer examples, but the related general formulas for all angles having the same sine, cosine or tangent are to be commended. The small pink representations of coordinate paper are attractive to the eye, but the graphs of