on any set E and

$$\int_E dx = \overline{\text{meas.}}\, E.$$

It is questionable *whether this precise formula is a decided improvement over M. Lebesgue's statement.* But, further, from this formula it is deduced that *the Pierpont integral does not enjoy the fundamental property that if E, F are sets with no points in common*

$$\int_{E+F} f(x)dx = \int_E f(x)dx + \int_F f(x)dx$$

(which however is true when E, F are "separated," according to Professor Pierpont). It suffices to apply this formula when $f(x) \equiv 1$, $E + F$ is an interval and E is non-measurable.

A REPLY TO A REPLY.

BY PROFESSOR JAMES PIERPONT.

As I view the issue between Professor Fréchet and myself, it may be summed up as follows:

1°. Professor Fréchet thought that it was possible to split a measurable set into two *separated* non-measurable sets, and he gave an alleged example. Since no such division is possible this example proved to be an ignis fatuus.

2°. Supported by this example, it was easy for Professor Fréchet to bring a number of grave charges against my work, in fact it might seem as if my whole theory had toppled to the ground.

3°. Professor Fréchet now admits (provisionally) that he was in error on this score, but he still holds to his "original assertion" that my integral definition "is inappropriate," "though for partly different reasons." What are these new reasons? Although I have read and reread the above article I have found but one, viz.: Suppose A is *non*-measurable and suppose B and C form a *non*-separated division of A, then the relation

$$\int_A = \int_B + \int_C \tag{1}$$

may not hold.

This very obvious fact I have known from the start; it is one that any one would discover. To Professor Fréchet this may be an insuperable objection and I have no contention with any one who holds this view. A similar peculiarity is presented in many theories. For example, one may take the stand that the double series

$$\begin{aligned} &a_{11} + a_{12} + a_{13} + \cdots \\ &+ a_{21} + a_{22} + \cdots \\ &+ a_{31} + \cdots \\ &\cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \end{aligned} \tag{2}$$

is convergent only when it is absolutely convergent,* but many authorities do not. In the latter case one is led to a number of unexpected results; e. g., the series (2) may converge although every series formed of a row or a column of (2) is divergent.

To my mind it does not seem wise to be doctrinaire in such matters. The relation (1) does hold for separated divisions of A, and when A is measurable no other divisions are possible. Since no one as yet has exhibited a non-measurable set, only the existence of such sets having been established, it seems at least premature to argue on a priori grounds against any theory which makes a step in advance.

In any case the nature of Professor Fréchet's objections has been widely changed; as first formulated they struck at the very foundation of my theory by impeaching the correctness of one of its main theorems; at present the only objection I see is an expression of a personal opinion.

4°. Polemics are apt to be interminable; fresh charges are made, fresh rejoinders necessitated and so on ad infinitum. I therefore am not astonished that Professor Fréchet has injected a new element into the discussion. It now seems that the vital point is the "real difference" between Lebesgue's definition and my own. To me this question is one of complete indifference and I leave Professor Fréchet to settle it entirely to his own satisfaction.

* Cf. C. Jordan, Cours d'Analyse, vol. I, p. 302.