The important thing here is fundamental to the whole question of notation and particularly to notational interchangeability. The rule of differentiation in situ and the ordinary rules for the use of dot and cross in vector algebra taken with the identity $\int dS() = -\int d\tau()$ suffice to prove all Dr. Poor's theorems and many others of the sort without reference to any list of formulas—the whole thing has become mere formal operation which for a student of Hamilton, Tait, Gibbs, and McAulay is in the same category as the work

$$a - \frac{1}{a} = \frac{a^2 - 1}{a} = \frac{(a+1)(a-1)}{a}$$

is for the schoolboy.* If this is equally true of the student of Burali-Forti and Marcolongo, I am both surprised and happy.

ON PIERPONT'S INTEGRAL. REPLY TO PROFESSOR PIERPONT.

BY PROFESSOR MAURICE FRÉCHET.

MY single aim in my previous contribution to this journal ("On Pierpont's definition of integrals," volume 22, number 6, March, 1916) was to point out that, in my own words, *this new definition is inappropriate. I still hold to my original assertion* (though for partly different reasons) and will show why I do so.

Thus the question whether two non-measurable sets with no points in common are separated or not is far from being the vital point. This being explicitly stated, I hasten to say that *concerning this last particular question,* Professor Pierpont is entirely justified in saying: "Professor Fréchet has been misled at this point . . . and his example establishes not an error on my part but a carelessness of reasoning on his." As a matter of fact, I too quickly assimilated in my mind "separated" with "having no point in common." The same thing occurred with the word "exterior" and my objection to theorem 341, page 346 arose from a miscon-

* It would not have been obvious to the schoolboy, perhaps not even to a professional mathematician, in the days before a suitable notation for elementary algebra had been developed.

ception of the meaning of this word in Pierpont's terminology. Indeed, as I was at the front (where I am still) when I wrote this article, I could only compose it out of notes formerly taken. And even now, I do not understand what Professor Pierpont describes as "sets exterior to each other." I will then let fall my objections to formulas (2), (5), (7), in Professor Pierpont's reply.

However, under present circumstances (I write this on June 30) it is wiser not to postpone my answer. And with but the information I have in hand, I will make good my point as follows:

The main differences between Lebesgue and Pierpont integrals are two in number.

I. When E is measurable and $f(x)$ is summable the common value of Lebesgue and Pierpont integrals $\int_E f(x)dx$ is arrived at in different manners.

Then, I still maintain that in this case the real difference between their definitions is *not* that—as Professor Pierpont asserts—he makes use of an infinite instead of a finite number of parts δ_i of E (as in Riemann's definition). It lies essentially in the use of measurable sets instead of intervals. For, in the most important case: when $f(x)$ is bounded, the finiteness or infiniteness of the number of parts δ_i of E is indifferent in Pierpont's definition. This is easily seen, starting from Pierpont's theorem, that the remainder

$$\overline{\text{meas.}}\ \delta_i + \overline{\text{meas.}}\ \delta_{i+1} + \cdots$$

(of a series equal to $\overline{\text{meas.}}\ E$) converges to zero.

It is open to Professor Pierpont to prove that this former assertion of mine is wrong.

I mention in passing that apropos of a different memoir, M. Lebesgue kindly pointed out to me that a definition of Lebesgue's integrals by means of Riemann's sums was given as early as 1905 by W. H. Young (*Philosophical Transactions*).

II. Pierpont's definition enables him to give a definite value to integrals which are not integrable according to M. Lebesgue.

Whereas when E is non-measurable, M. Lebesgue contents himself with saying that the measure of E is contained between $\overline{\text{meas.}}\ E$ and $\underline{\text{meas.}}\ E$, Professor Pierpont goes further. According to his definition, the function $f(x) \equiv 1$ is integrable on any set E and

$$\int_E dx = \overline{\text{meas.}}\ E.$$

It is questionable *whether this precise formula is a decided improvement over M. Lebesgue's statement.* But, further, from this formula it is deduced that *the Pierpont integral does not enjoy the fundamental property that if E, F are sets with no points in common*

$$\int_{E+F} f(x)dx = \int_E f(x)dx + \int_F f(x)dx$$

(which however is true when E, F are "separated," according to Professor Pierpont). It suffices to apply this formula when $f(x) \equiv 1$, $E + F$ is an interval and E is non-measurable.

A REPLY TO A REPLY.

BY PROFESSOR JAMES PIERPONT.

As I view the issue between Professor Fréchet and myself, it may be summed up as follows:

1°. Professor Fréchet thought that it was possible to split a measurable set into two *separated* non-measurable sets, and he gave an alleged example. Since no such division is possible this example proved to be an ignis fatuus.

2°. Supported by this example, it was easy for Professor Fréchet to bring a number of grave charges against my work, in fact it might seem as if my whole theory had toppled to the ground.

3°. Professor Fréchet now admits (provisionally) that he was in error on this score, but he still holds to his "original assertion" that my integral definition "is inappropriate," "though for partly different reasons." What are these new reasons? Although I have read and reread the above article I have found but one, viz.: Suppose A is *non*-measurable and suppose B and C form a *non*-separated division of A, then the relation

$$\int_A = \int_B + \int_C \qquad (1)$$

may not hold.