THE OCTOBER MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and seventy-ninth regular meeting of the Society was held in New York City on Saturday, October 30, 1915. The attendance at the morning and afternoon sessions included the following fifty members:

Professor R. C. Archibald, Dr. A. A. Bennett, Mr. D. R. Belcher, Professor E. G. Bill, Professor W. J. Berry, Professor G. D. Birkhoff, Professor E. W. Brown, Dr. T. H. Brown, Dr. Emily Coddington, Professor F. N. Cole, Professor Elizabeth B. Cowley, Professor Louise D. Cummings, Dr. H. B. Curtis, Mrs. E. B. Davis, Dr. C. R. Dines, Professor L. P. Eisenhart, Professor H. B. Fine, Dr. C. A. Fischer, Professor W. B. Fite, Professor O. E. Glenn, Dr. G. M. Green, Professor C. N. Haskins, Professor H. E. Hawkes, Professor L. A. Howland, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Mr. P. H. Linehan, Dr. H. F. MacNeish, Dr. L. C. Mathewson; Professor F. M. Morgan; Mr. George Paaswell, Dr. G. A. Pfeiffer, Professor H. W. Reddick, Professor L. W. Reid, Professor R. G. D. Richardson, Mr. J. F. Ritt, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor D. E. Smith, Professor P. F. Smith, Professor W. B. Stone, Mr. H. S. Vandiver, Professor Oswald Veblen, Dr. Mary E. Wells, Professor H. S. White, Miss E. C. Williams, Professor A. H. Wilson, Professor J. W. Young.

The President of the Society, Professor E. W. Brown, occupied the chair, being relieved by Vice-President Oswald Veblen. The Council announced the election of the following persons to membership in the Society: Mr. D. R. Belcher, Columbia University; Professor J. W. Calhoun, University of Texas; Professor Sarah E. Cronin, State University of Iowa; Mr. C. A. Epperson, State Normal School, Kirksville, Mo.; Dr. Olive C. Hazlett, Radcliffe College; Mr. C. M. Hebbert, University of Illinois; Miss Goldie P. Horton, University of Texas; Professor W. S. Lake, Bendigo School of Mines and Industries, Australia; Mr. D. H. Leavens, College of Yale in China; Mr. C. T. Levy, University of California; Dr. F. W. Reed, University of Illinois; Professor L. H. Rice, Syracuse University; Mr. J. F. Ritt, Columbia University; Professor

D. M. Y. Sommerville, Victoria University College, Wellington, N. Z.; Miss L. R. Stoughton, Rosemary Hall School, Greenwich, Conn.; Dr. C. E. Wilder, Pennsylvania State College; Mr. A. R. Williams, University of California; Dr. L. T. Wilson, University of Illinois; Dr. F. E. Wright, U. S. Geological Survey. Four applications for membership in the Society were received.

The Council submitted a list of nominations for officers and other members of the Council, to be placed on the official ballot for the annual election. A committee was appointed to audit the accounts of the Treasurer for the current year.

The dinner in the evening, always a pleasant feature of the meetings, was attended by twenty-one members and friends.

The following papers were read at this meeting:

(1) Dr. G. A. PFEIFFER: "Existence of divergent solutions of the functional equations $\varphi[g(x)] = a\varphi(x)$, $f[f(x)] = g(x)$, where $g(x)$ is a given analytic function, in the irrational case."

(2) Professor C. N. HASKINS: "On the extremes of bounded summable functions and the distribution of their functional values."

(3) Dr. G. M. GREEN: "Projective differential geometry of one-parameter families of space curves, and conjugate nets on a curved surface. Second memoir."

(4) Dr. G. M. GREEN: "The linear dependence of functions of several variables."

(5) Mr. A. R. SCHWEITZER: "On the dependence of algebraic equations upon quasi-transitiveness."

(6) Professor H. S. CARSLAW: "A trigonometrical sum and the Gibbs phenomenon in Fourier's series."

(7) Professor W. F. OSGOOD: "On a sufficient condition for a non-essential singularity of a function of several complex variables."

(8) Dr. DUNHAM JACKSON: "Singular points of functions of several complex variables."

(9) Professor W. F. OSGOOD: "On functions of several complex variables."

(10) Professor L. P. EISENHART: "Envelopes of rolling and transformations of Ribaucour."

(11) Professor W. B. FITE: "Note on linear homogeneous differential equations of the second order."

(12) Mr. H. S. VANDIVER: "Note on the distribution of quadratic residues."

(13) Professor G. D. BIRKHOFF: "A theorem concerning the singular points of ordinary linear differential equations."

(14) Professor H. S. WHITE: "Closed systems of sevens in a 3–3 correspondence."

(15) Professor W. R. LONGLEY: "Note on a theorem on envelopes."

(16) Mr. A. R. SCHWEITZER: "On the dependence of algebraic equations upon quasi-transitiveness. Second paper."

(17) Mr. A. R. SCHWEITZER: "A new functional characterization of the arithmetic mean."

Professor Carslaw's paper was communicated to the Society through Professor Bôcher. In the absence of the authors Professor Osgood's second paper was read by Professor Birkhoff and the papers of Mr. Schweitzer, Professor Carslaw, Professor Longley, and the first paper of Professor Osgood were read by title.

Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. The first functional equation considered in Dr. Pfeiffer's paper arises in the consideration of the conformal equivalence of curvilinear angles in the so called irrational case. In this case a transformation can always be found formally. The author shows that the series giving the transformation may be divergent. The second functional equation here considered determines the bisectors of an irrational curvilinear angle as defined by Kasner, and it is proved that such an angle may have two, one, or no bisectors.

Stated specifically the theorems obtained are

1. There exists an analytic function $g(x) \equiv a_1x + a_2x^2 + \cdots$ ($|a_1| = 1$, $a_1^n \neq 1$, $n = 1, 2, \cdots$) such that every formal solution $\phi(x) = c_1x + c_2x^2 + \cdots$ $(c_1 \neq 0)$ of the functional equation $\phi[g(x)] = a_1\phi(x)$ is divergent for all values of $x \neq 0$.

2. Given the functional equation $f[f(x)] = g(x)$. Then $g(x) \equiv a_1x + a_2x^2 + \cdots$ ($|a_1| = 1$, $a_1^n \neq 1$, $n = 1, 2, \cdots$) may be taken such that the given equation has (*a*) no solution analytic about the origin, (*b*) only one such solution, (*c*) two such solutions. The number of solutions cannot be greater than two.

The author is indebted to Professor Birkhoff for suggestions which led to the above development.

2. Professor Haskins's paper presents, first, a method of the integral calculus, as contrasted with the classic methods of the differential calculus, for the determination of the extrema extremorum of bounded Lebesgue-integrable functions; and second, certain results concerning what may be termed the statistical distribution of functional values.

The methods used are allied to those of Laplace-Darboux-Stieltjes for the determination of " functions of large numbers." Some of the results (for the case of continuous functions and Riemann integrals) have already been presented to the Society. The full significance of the results appears however only when the Lebesgue integral is used, for the reason that these results relate to certain constants and certain analytic functions which serve to divide all Lebesgue-integrable functions into classes such that all the functions of each class and only those have the same defining elements of their Lebesgue integrals.

3. In a recent paper,* Dr. Green laid the foundations for a purely projective theory of conjugate nets on a curved surface, his present paper forming a continuation of this study. The first part of the memoir contains a canonical development of the non-homogeneous coordinates of the surface in the neighborhood of a point, which was the subject of a previous communication to the Society.† The discussion is completed in the present paper. The tetrahedron of reference which gave rise to the canonical development has as two of its edges lines which Wilczynski‡ has called the axis and the ray of a point of the surface. The second part of Dr. Green's paper contains an investigation of the properties of the conjugate net expressed in terms of the axis and ray congruences. A new net of curves is introduced, viz., the associate conjugate net, the tangents to the two curves of which at any point separate harmonically the tangents to the curves of the original conjugate net. Any conjugate net has one and only one associate conjugate net. It appears that the theory of a conjugate net is very conveniently investigated through the consideration thereof in connection with its associate conjugate net. A discussion from this point of view leads to many interesting theorems, some of

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* *Amer. Jour. of Mathematics*, vol. 37 (1915), pp. 215–246.

† See this BULLETIN, vol. 20, no. 8 (May, 1914), p. 397.

‡ *Transactions*, vol. 16 (1915), pp. 311–327.

them already known; in fact, the study of a conjugate net, its associate conjugate net, and the axis and ray congruences for both, seems to afford a powerful method for describing geometrically projective properties of the conjugate net. An application is made at the end of the paper to the conjugate nets which Bianchi has called isothermally conjugate. For such a net, for instance, the associate conjugate net is also isothermally conjugate.

4. For a set of functions of a single real variable, it is well known that in certain cases the vanishing of the wronskian is a sufficient condition for linear dependence. The wronskian arises naturally out of the study of a single ordinary homogeneous linear differential equation, the characteristic property of which is that any solution is expressible linearly, with constant coefficients, in terms of a fundamental set of solutions. The natural generalization of this kind of ordinary differential equation is the completely integrable system of homogeneous linear partial differential equations, the solutions of which have the same property. A year ago, Dr. Green* gave a generalization for such systems of the notion of wronskian, but this can be done only when the completely integrable system is given. It is the object of Dr. Green's second paper to supply a sufficient condition for the linear dependence of a set of functions of several independent real variables. There is no determinant of fixed form analogous to the wronskian, but the criteria given are very general, including by specialization those concerning the wronskian for functions of a single variable. Application of the theorems is made to completely integrable systems, yielding results analogous to those for ordinary homogeneous linear differential equations.

5. In the first part of Mr. Schweitzer's paper it is shown that if

$$(1)\quad f(x_1 + y_1, x_2 + y_2, \cdots, x_{n+1} + y_{n+1}) = f(x_1, x_2, \cdots, x_{n+1}) + f(y_1, y_2, \cdots, y_{n+1}),$$

$$(2)\quad f\{f(t_1, t_2, \cdots, t_n, x_1)f(t_1, t_2, \cdots, t_n, x_2) \cdots f(t_1, t_2, \cdots, t_n, x_{n+1})\} = f\left(\frac{x_2}{c_1}, \frac{x_3}{c_2}, \cdots, \frac{x_{n+1}}{c_n}, \frac{x_1}{c_{n+1}}\right),$$

* See this BULLETIN, vol. 21, no. 4 (Jan., 1915), p. 162.

then

$$f(x_1, x_2, \cdots, x_{n+1}) = \sum_{i=1}^{n} [c'_{n-i+1}\xi^{n-i+2} \cdot x_i] + \xi x_{n+1},$$

where

$$c'_{n-i+1} = c_i \cdot c_{i+1} \cdots c_n, \qquad \frac{1}{c_{n+1}} = c_1 \cdot c_2 \cdots c_n \cdot \xi^{n+1},$$

$$1 + \sum_{i=1}^{n} c'_{n-i+1} \cdot \xi^{n-i+1} = 0.$$

In the second part of the paper an alternative method of inducing algebraic equations of the nth degree is discussed. The latter treatment is based on the equation (1) in conjunction with functional equations derived from the relation

$$f\{f_1(t_1^{(1)}, t_2^{(1)}, \cdots, t_n^{(1)}, x_1), f_2(t_1^{(2)}, t_2^{(2)}, \cdots, t_n^{(2)}, x_2) \cdots f_{n+1}(t_1^{(n+1)}, t_2^{(n+1)}, \cdots, t_n^{(n+1)}, x_{n+1})\} = \psi(x_1, x_2, \cdots, x_{n+1})$$

(or analogous relations obtained by the homologous transposition of the x's on the left side) by substituting suitable x's for some or all of the t's and assuming the remaining t's (if any) with the same subscript to be identical. In the simplest cases the following theorems are valid:

I. $f\{f(x_3, x_1, t), f(x_1, x_2, t), f(x_2, x_3, t)\} = f(x_2, x_1, x_3)$ implies

$$f(x_1, x_2, x_3) = \frac{1}{2}x_1 + \frac{1 \pm \sqrt{13}}{4}x_2 - \frac{3 \pm \sqrt{13}}{4}x_3.$$

II. $f\{f(x_2, x_3, x_1), f(x_3, x_1, x_2), f(x_1, x_2, x_3)\} = f(x_2, x_3, x_1)$ implies

$$f(x_1, x_2, x_3) = \tfrac{1}{3}(2x_1 - cx_2 - c^2x_3) \quad \text{or} \quad \tfrac{1}{3}(x_1 + cx_2 + c^2x_3) \quad \text{or} \quad \tfrac{1}{3}(x_1 + x_2 + x_3),$$

where $1 + c + c^2 = 0$.

III. $f\{f(x_3, t, x_1)f(x_1, t, x_2)f(x_2, t, x_3)\} = f(x_2, x_1, x_3)$ implies $f(x_1, x_2, x_3) = -c^2x_1 - c^4x_2 + cx_3$, where $c^3 + c - 1 = 0$.

IV. $f\{f(t, x_3, x_1)f(t, x_1, x_2)f(t, x_2, x_3)\} = f(x_2, x_1, x_3)$ implies

$$f(x_1, x_2, x_3) = -\frac{1}{2}x_1 + \frac{1 \pm \sqrt{5}}{4}x_2 + \frac{1 \mp \sqrt{5}}{4}x_3$$

or $2x_1 - x_2 - x_3$.

In deriving the preceding results the operation of differentiation is not employed.

6. In Professor Carslaw's paper the Gibbs phenomenon in Fourier series is deduced from the behavior of the approximation curves for the Fourier series

$$2(\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \cdots). \tag{1}$$

In the interval $-\pi < x < \pi$, this series represents the function defined by the following equations:

$$f(-\pi) = f(\pi) = f(0) = 0, \qquad f(x) = -\tfrac{1}{2}\pi(-\pi < x < 0),$$
$$f(x) = \tfrac{1}{2}\pi(0 < x < \pi).$$

Bôcher, in his discussion of the Gibbs phenomenon (*Annals of Mathematics*, (2), volume 7, 1906, and *Crelle's Journal*, volume 144, 1914), uses the series

$$\sin x + \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x + \cdots. \tag{2}$$

He deals with the maxima and minima of $R_n(x)$, the remainder after n terms.

Gronwall (*Mathematische Annalen*, volume 72, 1912) discusses the question of the behavior of the maxima and minima of $S_n(x)$, the sum of n terms of the series (2), and deduces the Gibbs phenomenon from the properties of $S_n(x)$. But both his results and the method by which they are obtained are somewhat complicated.

In the present paper the behavior of the maxima and minima of $S_n(x)$ for the series (1) is examined. The properties of the approximation curves $y = S_n(x)$ are much simpler for (1) than for (2). From the results obtained, the Gibbs phenomenon for the series (1) follows immediately. The extension to the general case is simple.

7. Professor Osgood's first paper contains the following theorem. In order that a function $F(z_1, \cdots, z_n)$ have a non-essential singularity in a point $(z) = (a)$ it is sufficient:

(1) that $F(z_1, \cdots, z_n)$ be uniquely defined and analytic at all points of a certain neighborhood Σ of the point (a), which do not lie on a locus defined by an equation of the form $H(z_1, \cdots, z_n) = 0$, where $H(z_1, \cdots, z_n)$ is analytic in the point (a) and vanishes there, but does not vanish identically.

(2) that $F(z_1, \cdots, z_n)$ in general become infinite in those points of the above locus which lie in Σ, the excepted points being those which lie simultaneously on a second locus, $G(z_1, \cdots, z_n) = 0$, where $G(z_1, \cdots, z_n)$ is subject to the same conditions as $H(z_1, \cdots, z_n)$, and where furthermore G and H have no common factor in the point $(z) = (a)$.

The proof is given in the author's Funktionenthеorie, volume II, chapter 3, § 2.

8. It is a familiar property of functions of a single complex variable that if $f(z)$ has a pole at the point a, it can be written throughout the neighborhood of this point in the form $f(z) = \varphi(z)/(z - a)^m$, where m is a suitable positive integer and $\varphi(z)$ is a function of z which is analytic at a and does not vanish there. Dr. Jackson gives one of the possible generalizations of this theorem to functions of several variables, differing somewhat in its hypotheses from one which has been obtained by Professor Osgood. It is assumed that $f(z_1, z_2, \cdots, z_n)$ is analytic throughout the neighborhood of $(a_1, a_2, \cdots, a_n)$, except for singularities which satisfy a certain requirement of continuity in their distribution, and which are of such a nature that f is analytic except for poles when regarded as a function of z_1 alone; and it is shown that under these conditions f can be expressed as the quotient of two functions, each analytic in the neighborhood of $(a_1, a_2, \cdots, a_n)$. The denominator is of course no longer merely a power of a linear factor, in general, and the numerator may vanish at points of the neighborhood in question. The proof is closely related in method to well-known proofs of Weierstrass's theorem of factorization.

9. Let $x_i = \varphi_i(u_1, \cdots, u_{n-1})$, $i = 1, \cdots, n$, be a set of functions, each analytic at the origin $(u) = (0)$, and vanishing there. Will the points $(x_1, \cdots, x_n)$ of the locus thus defined lie on a hypersurface $\Omega(x_1, \cdots, x_n) = 0$, where Ω is analytic at the origin and vanishes there, but does not vanish identically? If $n = 2$, it is well known that the answer is affirmative. Professor Osgood shows by an example that, when $n > 2$, the corresponding theorem is false.

Let the functions $f_i(u_1, \cdots, u_n)$, $i = 1, \cdots, n$, be analytic at the point $(u) = (b)$, and let $f_i(b_1, \cdots, b_n) = a_i$. Let T be an arbitrary neighborhood of the point $(a) = (a_1, \cdots, a_n)$.

Finally, let the Jacobian of the functions f_i vanish identically. Then it is well known that there exists a point (a') of T and a function $\Omega(x_1, \cdots, x_n)$ analytic at (a') and not identically $= 0$, and furthermore such that, if x_i be replaced by f_i, Ω then goes over into a function of $(u_1, \cdots, u_n)$ which vanishes identically in these arguments. Can the point (a') be taken at (a)?

Professor Bliss has shown that it can when $n = 2$. It is shown in the present paper that when $n > 2$ this is not in general possible.

Thirdly, the tentative generalization of Weierstrass's theorem of factorization, concerning which Professor Osgood raised a query in the Madison Colloquium Lectures, page 185, is shown to be impossible when $n > 1$.

Finally, a function z of x and y defined by the equation $G(x, y, z) = 0$, where G is analytic at the origin and vanishes there, but does not vanish identically, may have a natural boundary in every neighborhood of the origin.

10. In a note published in volume 23 of the *Rendiconti dei Lincei* Bianchi has defined as an envelope of rolling the surface enveloped by a plane invariably fixed with respect to a surface S_0 as the latter rolls over an applicable surface S, which Bianchi calls the surface of support. He shows that, given any surface Σ, the problem of finding pairs of applicable surfaces S_0 and S such that Σ is an envelope of rolling as S_0 rolls over S reduces to the integration of a partial differential equation of the second order and to quadratures. Two surfaces are said to be in the relation of a transformation of Ribaucour when they constitute the envelope of a two-parameter family of spheres such that the lines of curvature correspond on the two surfaces, corresponding points being on the same sphere. Professor Eisenhart has shown that the necessary and sufficient condition that either surface of two so related be an envelope of rolling with the locus of centers of the spheres for surface of support is that the correspondence of the spherical representations of the lines of curvature of the two surfaces in the relation of a transformation of Ribaucour be conformal. This latter condition is satisfied in case these spherical representations are isothermal, and only in this case. Any surface with isothermal spherical representation of its lines of curvature admits of transforma-

tions of Ribaucour of this kind, as the author has shown in several former papers.

11. Professor Fite determines certain intervals within which a solution of the equation $y'' + py' + qy = 0$ and its derivative cannot both vanish. The argument is based upon the obvious fact that such a solution and its derivative cannot both vanish in any interval within which q is negative.

12. In the BULLETIN for November, 1915, Mr. Vandiver considered some theorems which he applies in the present note to problems regarding the distribution of quadratic residues for a prime modulus. Some special quadratic forms are also considered.

13. Professor Birkhoff proves that if $Y(x)$ is the matrix solution of a linear differential system with singular point of rank $q + 1$ at $x = \infty$, then, if $\varphi(x) - x$ vanishes to the qth order at $x = \infty$, the matrix $A(x)$ defined by the equation $Y(\varphi(x)) = A(x)Y(x)$ is composed of elements analytic at $x = \infty$. The converse is also proved. This note is to appear in the *Proceedings of the National Academy of Sciences*.

14. In a previous note Professor White had proved a property of seven points on a gauche cubic curve: that certain sets of seven planes determined by them osculate other cubic curves. In the present note it is established that such a set of seven points on the one curve and seven osculating planes of the other curve is variable while the curves remain fixed, every point of the first curve belonging to one such set of seven.

15. The theorem in question was published by Risley and MacDonald in the *Annals of Mathematics*, second series, volume 12, page 86, and gives certain criteria for the existence and non-existence of an envelope of the system of curves $y = f(x, \alpha)$ in the neighborhood of a point (x_0, y_0). In the present note Professor Longley points out that the conclusion of the theorem does not follow from the argument and gives some examples which show that it is impossible from the hypotheses of the theorem to draw any conclusion concerning the non-existence of an envelope in the neighborhood of the point in question.

16. In a previous communication Mr. Schweitzer defined a (symmetric) group of functional equations on $n + 1$ variables ($n = 1, 2, 3, \cdots$) viz.,

$$f\{f(t_1, t_2, \cdots, t_n, x_1), f(t_1, t_2, \cdots, t_n, x_2), \cdots f(t_1, t_2, \cdots, t_n, x_{n+1})\} = f\{x_{i_1}, x_{i_2}, \cdots, x_{i_{n+1}}\}.$$

In the present paper the group of equations is studied which results from the preceding group by replacing, in the above relation, the x_{i_k} by the functions $\alpha_k(x_{i_k})$ $[k=1, 2, \cdots, (n+1)]$. In particular, the relation corresponding to the substitution $[1, 2, \cdots, (n+1)]$ implies that there exists a function $\psi(x)$ such that

$$\psi\alpha_k(x) = m_k\psi(x) + p_k,$$

$$\psi f(x_1, x_2, \cdots, x_{n+1}) = \sum_{i=1}^{n} \frac{\xi^{n-i+2} \cdot \psi(x_i)}{m_i \cdot m_{i+1} \cdots m_n} + \xi\psi(x_{n+1}) + l,$$

where ξ, m_k, p_k, l are constants such that

$$\xi^{n+1} = m_1 \cdot m_2 \cdots m_n \cdot m_{n+1}, \quad \sum_{i=1}^{n} \frac{\xi^{n-i+2}}{m_i m_{i+1} \cdots m_n} + \xi = 0,$$

and

$$\sum_{i=1}^{n} \frac{\xi^{n-i+2} p_i}{m_i m_{i+1} \cdots m_n} + \xi p_{n+1} = 0.$$

By making special assumptions concerning the functions $\alpha_k(x)$ the function $\psi(x)$ can be particularized.

17. The postulates constructed by Mr. Schweitzer for the arithmetic mean are as follows:

1. $f\{x_1 + y_1, x_2 + y_2, \cdots, x_{n+1} + y_{n+1}\} = f(x_1, x_2, \cdots, x_{n+1}) + f(y_1, y_2, \cdots, y_{n+1}).$
2. $f(x_1, x_2, \cdots, x_{n+1}) = f(x_2, x_3, \cdots, x_{n+1}, x_1).$
3. $f\{f(x_1, x_2, \cdots, x_n, x_{n+1}), f(x_2, x_3, \cdots, x_{n+1}, x_1), \cdots f(x_{n+1}, x_1, x_2, \cdots, x_n)\} = f(x_1, x_2, \cdots, x_n, x_{n+1}).$

F. N. Cole,
Secretary.