CONCERNING ABSOLUTELY CONTINUOUS FUNCTIONS.

BY PROFESSOR M. B. PORTER.

IN a paper "Sulle funzioni integrali" published in 1905 in the *Atti della R. Accademia delle Scienze di Torino,* Vitali defined an important class of functions of limited variation to which he gave the name of *absolutely* continuous functions. He defines these functions as follows:

Let $F(x)$ be a finite function of the real variable x in an interval (a, b), where $a < b$, and let (α, β) be a partial interval of (a, b), $a \leqq \alpha < \beta \leqq b$. Call $F(\beta) - F(\alpha)$ the *increment* of $F(x)$ in (α, β). Call the sum of such increments, if it is finite and determinate, over a group of distinct (α, β)-intervals, the *increment of* $F(x)$ *in this group;* then, if for every $\sigma > 0$ there exists a $\mu > 0$ such that the modulus of the increment of $F(x)$ over every group of intervals of sum less than μ is less than σ, then $F(x)$ is said to be absolutely continuous in (a, b). Vitali then shows that $F(x)$ is a continuous function of limited variation, while continuous functions of limited variation are *not* all absolutely continuous, and establishes among others the following important

THEOREM. $F(x) - F(a) \equiv \int_a^x \Lambda F(x)dx$, *where* $\int \Lambda F$ *denotes the Lebesgue integral of one of the derivates of* $F(x)$*; and absolutely continuous functions are the only ones possessing this property.**

Lebesgue had already shown that the derivates of continuous functions of limited variation are summable and that in certain special cases the Lebesgue integral is the primitive function. Vitali's necessary and sufficient condition completes Lebesgue's theory in an important particular and shows that absolutely continuous functions constitute an important generalization of the class of analytic functions, and just as analytic functions can frequently be defined by general descriptive properties it is to be expected that such properties might exist for Vitali's functions. It is the purpose of this paper to show that

* For a proof of this theorem see Vallée Poussin's Cours d'Analyse, Tome 1, § 265, 3d edition.

THEOREM I: *A continuous function of limited variation, whose derivates are infinite only over a denumerable point set E, is absolutely continuous.*

We first remark that in virtue of a theorem of W. H. Young* the set E will always be either denumerable or of the power of the continuum. The inverse theorem is not true, for we shall show, by means of examples, that absolutely continuous functions exist with infinite derivates over any assigned point set of measure zero.

To prove Theorem I, first consider the case where $F(x)$ is monotone increasing. Then†

$$‡\Lambda F(x) \geqq \Lambda \int_a^x \Lambda F(x) dx,$$

so that $\Lambda \int_a^x \Lambda F(x) dx$ is finite whenever $\Lambda F(x)$ is finite. We have now but to apply Vallée Poussin's generalization of Scheeffer's theorem (page 101, ibid.) to see that

$$\int^x \Lambda F(x) dx \equiv F(x) - F(a),$$

which proves that $F(x)$ is absolutely continuous.

To prove the general case we have but to note that

$$\int_a^x | \Lambda F(x) | dx \geqq \int_a^x \Lambda F(x) dx \geqq - \int^x | \Lambda F(x) | dx$$

and again apply Scheeffer's theorem.

As a corollary, we have that if the derivates become infinite over a reducible set, $F(x)$ is absolutely continuous.

We shall now show that no further generalization is possible. To do this consider the function§ $\phi(x)$ defined as follows:

Starting with any null set E and a number c, we take a set of intervals α_{ij} such that

$$\beta_i = \sum_{j=1}^{j=\infty} \alpha_{ij} \leqq \frac{c}{2^i},$$

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* *Arkiv för Matematik, Astronomi och Fysik,* vol. 1, Stockholm, 1903, or see Hobson's Theory of Functions of a Real Variable, p. 285, for an account of Young's work.

† Cf. Vallée Poussin: Cours d'Analyse, vol. 1, p. 275.

‡ ΛF denotes the upper right-hand derivate of $F(x)$.

§ See Vallée Poussin, loc. cit., vol. 1, p. 100, bottom.

so that

$$\sum_1^\infty \beta_i \leqq c.$$

The points of E are now enclosed in the open intervals α_{ij}, so that each point is inside of an infinite number of intervals, and $\phi(x)$ is defined to be the sum of all the α-intervals or parts thereof which lie to the left of x.

Thus $\phi(x)$ is monotone and can easily be shown to be absolutely continuous as follows:

If $i + j = N$ is chosen sufficiently large, the $\phi_N(x)$ formed for this finite set of intervals will be absolutely continuous and as near as we please to $\phi(x)$ for all values of x. Hence $\phi(x)$ is absolutely continuous.

If the set E is not an *inner limiting* set, the set $E'' = E + \bar{E}$, which lies inside an infinite number of α intervals, will be such a set, and $\phi(x)$ will have an infinite derivative at all the points of E'' and no others. The set E may itself be an inner limiting set, in which case $\bar{E} = 0$.

It would be interesting to determine whether all absolutely continuous functions are of the form

$$F(x) + \phi(x),$$

where $F(x)$ has limited derivates.

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ON THE REPRESENTATION OF NUMBERS IN THE FORM $x^3 + y^3 + z^3 - 3xyz$.

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IF by $g(x, y, z)$ we denote the form

$$(1) \quad \begin{aligned} g(x, y, z) &= x^3 + y^3 + z^3 - 3xyz \\ &= (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx), \end{aligned}$$

then it is well known that

$$g(x, y, z) \cdot g(u, v, w) = g(xu + yw + zv,\ xv + yu + zw,\ xw + yv + zu).$$