rapport a été entièrement brûlé le 27 août dernier à Louvain, avec tout le magasin de mon éditeur, le 3e jour de l'incendie de cette ville par l'armée allemande.

Je vous prie, cher collègue, d'agréer l'expression de mes remerciments et l'assurance de mon entier dévouement,

C. DE LA VALLÉE POUSSIN.

CAMBRIDGE, MASS.,
May 11, 1915.

The objection to the formulation of Scheeffer's theorem referred to in Schoenflies' Bericht, volume 2, page 317, was directed at Scheeffer's statement of it. The reviewer was under the impression that even as stated above the *peut-être* might be objectively interpreted.

THE REVIEWER.

ENUMERATIVE GEOMETRY.

Lehrbuch der abzählenden Methoden der Geometrie. By H. G. ZEUTHEN. Leipzig, Teubner, 1914. xii + 394 pp. Price (cloth) 17 Marks.

IN the preface to his Lehrbuch Zeuthen expresses his gratitude to the publishers, "that the researches, which I have delighted to pursue from youth to an advanced age, may now appear in their full sequence." The mathematical world also has reason for hearty gratitude, not only to the Teubner firm, of whose family of publications this book is a very worthy member, but much more to Zeuthen himself, that he has produced a book summing up most carefully and elegantly both the chief results and the most fertile methods of enumerative geometry.

Zeuthen must have wished more than once in writing this work that a book were not essentially a one-dimensional configuration. The greater part of the book could have been displayed most satisfactorily in a plane with an axis of methods perpendicular to an axis of subjects. This arrangement being impossible, the author chose to make his work primarily a text on methods, and so to devote each chapter to a single method or group of methods. Within each chapter the results are grouped according to the configurations to which they apply, usually in the following order: plane curves, surfaces (in S_3), space curves, line configurations. The defect of one-dimensionality is then remedied by a short table of contents at the end, where all sections on the same figure are grouped together, and by a very complete set of cross-references in the text. We have, then, a treatise which may be regarded either as an exposition of the enumerative methods of projective geometry or as a very extensive account of the results obtained by those methods in the case of the most important types of figures in three-dimensional space.

The book is essentially one on projective geometry. To be sure, some of the apparatus of the geometry of birational transformations is developed. The genus of a curve, the arithmetic genus and the Zeuthen-Segre invariant of a surface are defined; but both definitions and applications are in terms of projective characteristics of the figures. On the other hand, the author descends to affine and metric geometry with readiness; a quite unexceptionable procedure, especially since he draws attention to the fact that these are but special cases of projective geometry.

The enumerative methods described by Zeuthen are applied almost exclusively to figures defined by algebraic equations. In a few places, to be sure, it is remarked that a method is applicable to systems of curves defined by means of certain algebraic differential equations—as (Article 163) in determining the number of curves of a system having contact of given order with a given curve; but even there the problem to be solved originally is purely algebraic.

A problem of enumerative geometry is one which asks the number of points, lines, curves, etc., of a system which fulfil certain conditions. Thus the number of intersections of two plane curves of given order can be regarded either as the number of the ∞^2 points in the plane which lie on both curves, or as the number of the ∞^1 points of one curve which lie on the other. Every such problem can be defined also as a determination of the number of solutions, that is, of the degree, of an algebraic equation.

The fundamental principle of enumerative geometry is the law of the "preservation of the number" (Erhaltung der Anzahl). That was stated by H. Schubert* in the following form: Let there be a variety of ∞^n objects on Γ, which shall be imposed a condition of dimension n, defined by assigned relations between Γ and another variety Γ'. Then the (finite)

* H. Schubert, Kalkül der abzählenden Geometrie (1879).

number of objects Γ satisfying that condition remains unaltered, however we may particularize Γ', provided the number remains finite. There arise a very considerable number of difficulties and dangers in the use of the methods based on this principle, and the care with which Zeuthen has treated them is worthy of all praise.

At the outset he lays down three necessary precautionary rules, which are never lost sight of. The first states that every case of a general problem must be considered as a limit of a continuous series of cases. An obvious corollary of this is expressed later in these words (Article 158): "In the application of formulas which express the number of configurations fulfilling diverse conditions, one must retain for each condition the exact meaning which was laid down at its introduction into the formulas." Thus, an arbitrary line through a double point of a plane curve is to be considered as a tangent to the curve if a tangent is defined as a line which has two coincident intersections with it; this is confirmed by the fact that such a line is the limit of a tangent to a curve which "acquires" a double point. On the other hand, it is not to be taken as a tangent if, the curve being defined by an equation in line coordinates, a tangent is regarded as an element of it. The second rule is that, when coincident solutions of a problem are counted, care must be taken that such solutions are counted the due number of times. Thus, the line from an arbitrary point of the plane to any double point of a plane curve—if we take the first definition of a tangent—counts as two tangents; this again would seem natural from the aspect of a curve about to acquire a double point. The third rule (this one expressly included in Schubert's enunciation) is that an enumerative formula loses validity and significance if the objects which it normally enumerates turn out, in a particular case, to be infinite in number. Thus, the unique value of dy/dx usually proves the existence of a unique tangent at a point of a curve; at a singular point dy/dx assumes the form $0/0$, and there is a tangent through the point in every direction—if we define a tangent rightly.

Study and Kohn* remarked in 1903 that the principle of the preservation of the number, as stated by Schubert, is not

* Study, Geometrie der Dynamen, p. 378. Kohn, "Ueber das Princip von der Erhaltung der Anzahl," *Archiv der Mathematik und Physik* (3), Bd. 4, pp. 312–316.

always valid; Kohn, indeed, said that it was "in a certain sense incurably ill." That is, a condition expressed in general form may be fulfilled in a certain finite number of cases; whereas, for certain particularizations of these conditions, the number of solutions, though remaining finite, may be increased. The source of this phenomenon is the possibility that the conditions may be fulfilled in different ways. The particular problem examined by Study and, in his article on the principle of the preservation of the number, by Severi,* is that which asks the number of projectivities of a line which transform into itself a given group of four points. If the cross-ratio of the group is not a cube root of -1, the number of projectivities is 4. If the cross-ratio is -1, there are 8 solutions; and if it is an imaginary cube root of that number, there are 12. The excess comes, as Severi remarks, from the possibility, which exists for harmonic and equianharmonic groups alone, that a projectivity may leave unchanged some of the points and interchange the others.

Severi stated and proved a theorem giving the conditions under which Schubert's principle can be safely applied. His briefer enunciation of it is this: "The principle of Schubert holds only for those conditions which can be resolved into sums of irreducible conditions of the same dimension." Thus, in the example which Severi cites from Study, the condition that a group of four points of a line be left invariant by a projectivity is the sum of four conditions: (α) that a projectivity be involutory and interchange all of a group of four points, (β) that it be involutory and permute two points of a group, not moving the others, (γ) that it be cyclic of order 3 and permute a group, (δ) that it be cyclic of order 4 and permute a group. (β), (γ), (δ) are of higher dimension than (α). Thus Severi amputated the incurable member, and left us the certainty that the body, after the operation, was quite free from disease. It was a beautiful and valuable piece of work.

And yet Zeuthen, while he says expressly (Article 189) that he has read Severi's article, does not quote his theorem. Severi's restriction of the availability of Schubert's principle achieves no more than the precautions which Zeuthen teaches. Consider, in particular, the problem examined by Severi.

* Severi, "Sul principio della conservazione del numero," *Rendiconti del Circolo Matematico di Palermo*, vol. 33 (1912), pp. 313–327.

The projectivities of the line which transform harmonic and equianharmonic groups into themselves, while interchanging only some of their points, can not be the limits of projectivities which transform into themselves groups with cross-ratios approaching a cube root of -1; for the latter transformations interchange either all or none of the points of the groups. The projectivities applicable to groups of special cross-ratios only are, then, by Zeuthen's first rule, expressly excluded from those enumerated. It seems possible, then, to do without Severi's theorem; yet that theorem is so elegant, clear, and simple, that it will prove a great aid to workers in enumerative geometry, and might well have been for that reason quoted in the present work.

An interesting and practically valuable use of reasoning from a particular to a more general case is offered by Zeuthen's justification of deductions made from real figures. He reminds us (Article 47) that all possible cases of a theorem depend on the values of a certain number of parameters, and that a real figure represents any one of an infinite number of sets of values of those parameters—all within certain limits. Enumerative properties observed in the figure, holding as they do for an infinite number of values of the parameters, will hold for all values—even such as make the figure imaginary. In this manner he obtains the fact that if the points of tangency of a single branch of a curve with a double tangent approach each other, two points of inflection also approach each other, and all four coincide where the curve acquires four-point contact with the line. That some caution is needed in reasoning from figures is shown by consideration of the number of inflections absorbed in a double point. The figure shows that a curve about to have a double point has at least two inflections which approach that point; analysis proves, however, that there are in truth six, four of which must be imaginary.

So much for examples of the care which Zeuthen exercises and inculcates,—surely the most important quality in a treatise on his subject. Praise should also be bestowed on the almost universal clearness of exposition. If exceptions are noted, let it be remembered that exceptions are far to seek. The first sentence of Article 32 reads thus: "Da die Komplexe einer gegebenen Ordnung m eine zusammenhängende Menge bilden, kann man einen solchen in abzählenden Untersuchungen spezialisieren, z. B. in der Weise, dass man verlangt, die Strahlen sollen m gegebene Gerade schneiden." On its face, this demands that each ray cut all of m given lines; while the meaning is, of course, that it cut some one of them. Another inaccurate passage occurs in the third paragraph of Article 172. The system there spoken of should be determined by one line and three points.

The references to sources are very few. For the most part, Zeuthen contents himself with mention of the bibliography in his article in the German Encyclopedia (III C 3) on the same subject. It is unfortunate that for the many developments of line geometry in the present treatise the Encyclopedia article has no mention.

Zeuthen's modesty gives another cause for regret. It would serve the reader better to become familiar with "Zeuthen's formula" than with the "allgemeiner Geschlechtsatz"; with the "Zeuthen-Segre invariant" than with "I."

A laudable feature of the book is the great number of exercises. They cover applications of all the principles developed and vary widely in difficulty—some being, as the author says, suitable even for doctor theses.

Chapter I is introductory. Its first half discusses the methods and aim of enumerative geometry, and gives definitions. The second half treats the method of determining the number of solutions falling together. An ingenious scheme, due to Zeuthen, for fixing this number appears in various forms in different sections of the book. Its first form is as follows: "The number of intersections of a line a with a curve at a point A can be defined as the sum of the orders of infinitesimal segments between a and the intersections of the curve with l, a straight line making a finite angle with a, at a distance from A that is infinitesimal of the first order." There follow a clever proof of Bézout's theorem and one of Halphen's theorem concerning the point multiplicity and line multiplicity of an element of a curve.

The first part of Chapter II deals with direct applications of the principle of preservation of the number, such as the determination of the enumerative properties of polar curves, of Hessians, of Reye's complex. By the use of Schubert's principle, theorems are deduced concerning general curves and surfaces from the consideration, as special cases, of degenerate ones. The degenerate plane curve of degree n

may be a set of n straight lines. It may be the projection of the curve on a line in its plane; that is, the line counted n times, with a certain number of vertices (Scheitel). In either case the simplification is so great as to create a highly efficient engine.

The next section treats problems with an excessive and hence infinite number of solutions. The problems solved concern largely the number of points necessary for determining completely various configurations,—plane curves, surfaces, curves on surfaces. Two paragraphs are devoted to Poncelet's "closing theorems" (Schliessungssätze), which deal with the question of whether polygons whose sides and vertices are respectively tangents and points of certain conics belonging to a single pencil are closed or open. Problems on the closing of polygons whose sides and vertices fulfil various conditions have been of unusual interest to Zeuthen; he treats them, indeed, in eight sections of the book.

The section entitled "Problems with no solutions" is largely devoted to applications of the principle that an algebraic function which is not constant must be able to assume all values. This principle is remarkably fertile; it furnishes, for instance, proofs of the constancy of various cross-ratios, among them that of lines from a variable point of a conic to four fixed points on it, and that of the tangents to a plane cubic from a variable point on it.

Chapter III is concerned with applications of Zeuthen's formula (allgemeiner Geschlechtsatz). If between the points of two curves c_1, c_2 of genus p_1 and p_2 respectively there exists an (α_1, α_2) correspondence; if, further, the number of cases in which two of the α_1 points corresponding to a point of c_2 coincide is η_1, and the inverse number is η_2, the formula is

$$\eta_2 - \eta_1 = 2\alpha_1(p_2 - 1) - 2\alpha_2(p_1 - 1).$$

An immediate consequence is Riemann's theorem of the equality of the genera of two curves in (1, 1) correspondence with one another. There is a careful discussion of Plücker's equations and of the analysis of more complicated singularities of plane curves. Plücker's equations, together with Zeuthen's formula, offer a means for investigating the order, class, and singularities of a curve in correspondence with a given curve (for example, its evolute). There follows a first consideration of systems of curves. The author treats the application of his and Plücker's formulas to space curves by means of their projections on a plane, and to surfaces by means of their circumscribed cones. A short section is given to the "Geschlechtsätze" for surfaces, and their application to curves on surfaces in (1, 1) correspondence. In the formulas in question the arithmetic genus and the Zeuthen-Segre invariant play the part taken by the genus of a plane curve in Zeuthen's formula.

The Cayley-Brill correspondence principle for points on a curve and analogous ones for points in a plane, on a surface, and in space, together with a wealth of applications of these theorems, form the subject matter of Chapter IV. In developing the Cayley-Brill theorem, Zeuthen defines the valence (Wertigkeit) k of an (α_1, α_2) correspondence between points of a curve of genus $p > 0$ by means of the formula

$$\gamma = \alpha_1 + \alpha_2 + 2kp,$$

γ being the number of self-corresponding points. He then proves that if the point P_1, taken k times, and the α_2 corresponding points P_2, each taken once, form the complete intersection of the ground-curve with a curve depending on P_1, then the value of k coincides with that obtained from the above equation. This order of development seems a little unnatural, but it has the great advantage of giving to negative and fractional valences equal rights with their more normal brothers. The final sections of this chapter are devoted to the correspondence principle for points on a surface, which Zeuthen announced in 1906. The form of this theorem is similar to that of Cayley-Brill, though naturally more complicated; the Zeuthen-Segre invariant takes the place, in a way, of the genus, and the valence has an exactly analogous interpretation. If one is to judge from the fruitfulness of its prototype, it should play an important part in the theory of surfaces.

The title of Chapter V is "Systems of Configurations." Systems of curves (in particular, of conics), of surfaces, and of correlations are treated; usually for the purpose of finding the number of elements of a system fulfilling given conditions. In the extended discussion of systems of conics, due attention and care are given to what one rather hesitates to speak of as Halphen's degeneracy. That is a conic which, in line coordinates, is a point, in point coordinates a line through the point; and which, considered as a limit of conics of a system, baffles description by both point and line coordinates.

The closing chapter deals with the powerful and elegant methods of Schubert's symbolic calculus. We could wish for a more complete statement of the meaning of the symbolism; yet the chapter is well written, and presents perhaps the most interesting method treated in the book. The work ends with a section which opens the way to the application of Schubert's methods to four-dimensional geometry.

The typography of the book is good, but not quite up to Teubner's highest standard—the standard, for instance, attained in Zeuthen's Encyclopedia article. I give a partial list of misprints discovered, with genuine regret at ending thus a review of a treatise so important for both the science and the art of mathematics.

Page 74	line 29, 30	*for* (ab)	*read* (ac)
79	1	*for* c_1	*read* c_2
102	5	*for* c_2	*read* c_n
112	4	*for* $\frac{1}{2}(n-1)(n-2)=d-e$	*read* $\frac{1}{2}(n-1)(n-2)-d-e$
125	12	*for* e	*read* c
139	28	*for* Fläche	*read* Kurve
145		*for formula* (1)	*read* (3)
146		*for formulæ* (2), (2′)	*read* (4), (4′)
163	10	*for* m_1'''	*read* m_4'''
208	37	*for* $n(k-1)$	*read* $k(n-1)$
250	9	*for* P_4	*read* P_7
276	35	*for* m'	*read* m''
286	1	*for* τ_0	*read* η_0
286	6	*for* γ_n	*read* γ_{0-n}
298	last	*for* c_1	*read* c_3
301	30	*for* γ_1	*read* p_1
315	14	*for* $\alpha\mu+\alpha\mu'$	*read* $\alpha\mu+\alpha'\mu'$
317	27	*for* (4)	*read* (2)
329	31	*for* einen Punkt und drei Gerade	*read* eine Gerade und drei Punkte
329	33	*for* auf einer der gebenen Geraden	*read* auf der gegebenen Geraden
334	27	*for* [175]	*read* [174]
336	24	*for* $(\mu\mu'^2)+4\beta$	*read* $(\mu\mu'^2)=4\beta$
384	23	*for* $\zeta g_p=\zeta_p+\zeta G$	*read* $\zeta g_p=\zeta p+\zeta G$

Brown University.

Edward S. Allen.