VALLÉE POUSSIN'S COURS D'ANALYSE.

Cours d'Analyse Infinitésimale. Par CH.-J. DE LA VALLÉE POUSSIN. Tome 1, troisième édition considérablement remaniée, et tome 2 remaniée. Louvain, Dieudonné, 1914. 9 + 452 pp. and 9 + 464 pp.

IN the two four hundred and fifty page volumes of this Cours the author has in mind two classes of readers. There are, first, those who desire to acquire an accurate working knowledge of the calculus stripped as far as possible of those subtilities which are repellant and useless to the engineer and physicist. This part of the book is printed in large type and follows in the choice of topics the general outline of the traditional French Cours, except that the space devoted to the treatment of Fourier's series is somewhat greater and convergence proofs are given. The handling throughout is clear, elegant, and concise; the various topics are illustrated by numerous carefully chosen examples selected with rare pedagogic skill to develop a real understanding of the text.

The rest of the Cours, printed in smaller type, is addressed to a different class of readers, those who wish to get at the fundamental principles of modern analysis. These last editions show that both volumes have undergone considerable alterations and improvements, proofs have been recast and expanded and the books, though excellent in the first edition, have been greatly improved.

§§ 8–10 deal with sets in general, and it would be hard to find anywhere so lucid and compact a presentation of the fundamental ideas involved. § 10 is concerned with the Borel-Lebesgue theory of measure and establishes the important results of Borel and Lebesgue, the methods of proof being essentially those later used by Vitali in his paper "Sui gruppi di punti" in volume 28 of the *Rendiconti del Circolo Matematico di Palermo.*

§ 12 deals with measurable functions and it is shown that practically all convergent processes applied to measurable functions lead to measurable functions.

§ 13 is concerned with functions of limited variation, destined later to play such an important rôle in the theory of Lebesgue integrals, and ends with a section on Vitali's *absolutely* continuous functions, the fundamental importance of which in Lebesgue's theory does not seem to be generally recognized.

The above constitutes an introduction to the subsequent treatment, which now begins with the elementary theory of derivatives followed in § 111 by arithmetic demonstrations of the familiar properties (due largely to Dini) of the four *derivates*. Here the book contains certain new matter, the most striking being the author's generalization of a famous theorem of Scheeffer concerning the determination of a continuous function when one of its derivates is known except for a null set.

The statement of this theorem is modeled somewhat after Scheeffer's, which has been criticized by Schoenflies as being illogical. We give it in the author's own words.

Si, dans un intervalle (a, b), *deux fonctions* $f_1(x)$ *et* $f_2(x)$ *ont leurs nombres dérivés supérieurs à droite*: 1° *finis en chaque point sauf peut-être dans un ensemble* E_1, *et* 2° *égaux sauf peut-être dans un ensemble de mesure nulle, les deux fonctions ne different que par une constante à moins que* E_1 *ne contienne un ensemble parfait.*

As stated, the theorem might seem self-contradictory for if $f_1 \equiv f_2 + c$ then the derivates will be *everywhere* equal.

This generalization of Scheeffer's theorem is not quite as general as it seems, for W. H. Young has shown that the set of points E_1 is either denumerable or has the power of the continuum, so that the theorem only holds when E_1 is denumerable. The generalized theorem admits, as the author points out, a sort of inverse, though in the proof given on page 102 the functions $y = \psi(x)$ and $x = \psi^{-1}(y)$ are not both continuous, as stated; one of them is not even singly valued.

The theory of Riemann integrability receives an elegant but very summary treatment and the author begins his exposition with the remark, "cette théorie n'a plus guère qu'une importance historique, car elle rentre comme cas particulier dans celle de Lebesgue, qui sera étudiée dans le chapitre suivant." A statement true only of proper Riemann integrals.

Chapter seven begins the systematic treatment of Lebesgue's integrals and under the general theory gives the relation to Riemann's integrals and six sufficient conditions for the validity of the equation

$$\int_a^b \lim_{n=\infty} f_n(x)dx = \lim_{n=\infty} \int_a^b f_n(x)dx,$$

wherein the greater simplicity of the conditions for Lebesgue integrals over those for Riemann's is amply evidenced. Summable functions are considered from the start and it is shown that, unlike the Riemann improper integral, the Lebesgue integral can always be defined as the limit of a sum $\Sigma l_i e_i$, and that the integral of a summable function has a derivative equal to the integrand except over a null set (presque partout—which we may translate "almost everywhere"). The treatment from now on shows a marked departure from that of Lebesgue in that it is less elementary but easier reading.

The author has devised a method of *majorating* ϕ_1 and *minorating* ϕ_2 functions such that

$$\phi_1 > \int_a^x f(x)dx > \phi_2$$

and

$$\Lambda\phi_1 > f(x) > \Lambda\phi_2$$

and with their aid establishes the capital theorem that the Λ of any monotone function f is summable and that its integral differs from $f(x)$ by a function $V(x)$ defined as the *variation* of $f(x)$ over the set of points E where $\Lambda f(x)$ is infinite.

Finally he establishes the corner stone of the theory by showing that the *necessary* and *sufficient* condition that

$$f(x) - f(a) \equiv \int_a^x \Lambda f(x)dx$$

is that $f(x)$ be *absolutely* continuous, as pointed out by Vitali in his paper in the *Atti della R. Accademia delle Scienze de Torino*, 1905, "Sulle funzioni integrali."*

Whether or not all functions with summable derivates belong to the class of functions of limited variation is left open, though it seems that this could have been answered in the affirmative from the theorems demonstrated in the text.

Original matter is taken up in § 267, where integration by substitution is considered. Here the results are of remarkable simplicity and generality, the final result being: If $f(x)$ is a

* The question of priority here is doubtful. Schoenflies, in the second volume of his Bericht, refers to papers by Levi of about the same date and does not mention Vitali, of whose papers he does not seem to be aware. A theorem of the author's in the first edition *practically* amounts to the condition of Vitali.

limited* summable function in (a, b) and $x = \phi(t)$ an absolutely continuous function of t such that x varies from $x_0 = \phi(t_0)$ to $X = \phi(T)$ always remaining in ab, then

$$\int_{x_0}^{X} f(x)dx = \int_{t_0}^{T} f(\phi)\phi'(t)dt,$$

where $\phi'(t)$ is defined to be zero in the null set where it does not exist.

The proof here could have been somewhat simplified if the author had made use of the absolutely continuous function $\phi(x)$ used at the top of page 101.

The chapter closes with an investigation of the properties of the second generalized derivates and derivatives. The proofs make liberal use of geometric intuition, though their arithmetization would probably not be difficult. Condition (K), § 274, is described in too summary a manner and it is not at once evident how it differs from merely postulating a right-handed derivative for the function $F(x)$. Finally it is shown that if $f(x)$ is summable and between (or equal) to the upper and lower right (left) generalized second derivates of $F(x)$,

$$F(x) \equiv \int_{a}^{x} dx \int_{o}^{x} f(x)dx + Lx + n.$$

These theorems play an important rôle in the theory of Fourier's series and the author has shown their power in his own researches to which reference will be made later.

In §§ 342 et seq. continuous and closed curves are treated and the author fills in the lacunæ of his proof that a closed continuous curve without double points divides the plane into two parts. Here certain topological theorems concerning *chains* play a leading rôle. A *link* is a connected region of the plane bounded by an uncrossed outer polygon and containing various *holes* bounded by polygons of the same sort; a *regular open chain* consists of a series of links such that *consecutive* and only consecutive links have points in common. The theorem to be established is that a closed continuous curve determines a sequence of thinner and thinner closed regular chains containing the curve in the links. The only

* The text has it *finite*, but the lemma on which the proof rests is not true unless the function is limited, as has been pointed out by Dr. Dunham Jackson.

postulate (not explicitly stated) needed to carry out the proof is that a closed uncrossed polygon of N sides divides the plane into two parts.

Rectifiable and quadrable curves are then taken up and the necessary and sufficient conditions are obtained together with the formula

$$s = \int_{t_1}^{t} \sqrt{x'^2 + y'^2}dt,$$

where the integral of Lebesgue is used and $x(t)$ and $y(t)$ are absolutely continuous functions of t, and the formulas for area

$$\int_{t_1}^{t} xy'dt, \quad -\int_{t_1}^{t} yx'dt, \quad \tfrac{1}{2}\int_{t_1}^{t} (xy' - yx')dt,$$

where the integrals are Lebesgue's and the only hypothesis is that the function (functions) whose derivatives figure shall be absolutely continuous.

This volume closes with three sections on *quasi-uniform* convergence—a name proposed by Borel to take the place of Arzelà's *convergenza uniforme a tratti*—and an elegant proof of Arzelà's celebrated theorem that *the necessary and sufficient condition that the limit of a convergent sequence of continuous functions be continuous is that the convergence be quasi-uniform.* Arzelà's necessary and sufficient condition for the termwise integrability of a series using Riemann integrals is not touched upon because the matter is so much simpler when Lebesgue integrals are used. The sufficient conditions in the latter theory are stated and proved.

A note supplementary to the second edition of volume two has been added dealing with the uniqueness of trigonometric developments,* where among others the following interesting theorem is proved:

If the coefficients of a trigonometric series approach zero with $1/n$ and the upper and lower limits of $S_n(x)$ for n infinite are summable and finite save in a null set E, the trigonometric series will be a Fourier series if E is not of the power of the continuum.

Here, as in the case of Scheeffer's theorem, the statement holds only in the case that E is a denumerable set.

In connection with this theorem it is of interest to note that

* Taken from two papers by the author in the *Bulletin de l'Académie royale de Belgique*, No. 11 (1912), No. 1 (1913).

Hugo Steinhaus has constructed a trigonometric series *everywhere* divergent where coefficients approach zero as limit. If this is a Fourier series, it is an example of the long sought everywhere divergent Fourier series.

The second volume is largely devoted to functions of several variables and after taking up double integrals from the more elementary standpoint proceeds to establish for the more advanced reader the leading theorems in the Riemann theory in the author's usual terse and elegant fashion, followed immediately by an extensive exposition of Lebesgue multiple integrals, where the theory follows, in a way, the broad outlines of functions of a single variable but where new concepts must be introduced, such as *density of a set in a point* and a generalized definition of derivatives. A theorem of Vitali's somewhat resembling the Heine-Borel theorem is then established and the important theorem:

An additive absolutely continuous function of a set has almost everywhere a finite and determinate derivative and is the indefinite integral of this derivative.

This is the analogue of the theorem already stated for absolutely continuous functions of a single variable.

The theorems of Lebesgue and Fubini on iterated double integrals follow and illustrate in a striking manner the greater simplicity and generality of the sufficient conditions in Lebesgue's theory over those in Riemann's.

The chapter closes with a generalization of Green's theorem where it is shown that

$$\int_C P dx + Q dy = \int\int_D (Q_x' - P_y') dx dy$$

provided

1°. that P and Q are continuous inside of C (over D).

2°. that P is absolutely continuous in y and Q absolutely continuous in x.

3°. P_y' and Q_x' are summable in D.

The author goes on to remark that the absolute continuity of P and Q would be secured if $\Lambda_y P$ and $\Lambda_x Q$ are finite. The *limitedness* of these derivatives is a sufficient condition for both 2° and 3°. Applied to the standard proof of Riemann of Cauchy's theorem that

$$\int_C f(z)dz = 0, \quad f(z) \equiv u(xy) + iv(xy)$$

when $f(z)$ is analytic inside C and on the boundary, this theorem shows that we need not even assume the *existence* of $f'(z)$ in D, but need only assume that u and v have *bornées* first derivatives satisfying the Cauchy-Riemann partial differential equations (cf. Goursat's well-known proof).

Next follows a beautiful chapter on the approximate representation of analytic functions, which closes with the treatment of Fourier's series in which the most important results of Dirichlet, Riemann, Dini, Cantor, Fejér, and Lebesgue are established.

In the compass of such a review, it is impossible to point out all the merits of these volumes, so rich in varied topics, so lucid in exposition and elegant in presentation. A unique feature of the book is that it does for Lebesgue's integrals what Jordan did for Riemann's theory.

Aside from his lectures delivered at the Collège de France under the Peccot foundation and published in 1904 in the Borel Series of Monographs on topics in the theory of functions, and his Lectures on Trigonometric Series, Lebesgue has published no systematic exposition of his ideas, contenting himself with the publication of numerous papers in various journals and transactions. The great value of the theories with which he has enriched analysis makes a systematic presentation of them a matter of great importance and we owe Professor Vallée Poussin a profound debt of gratitude not only for having completed this theory in many essential particulars but for his masterly presentation of it as a whole. With such a treatise available, these theories will become the common property of all mathematicians and, while certain simplifications and improvements in the demonstrations will come about in time, the outline and main structure has been definitely fixed. From the simple but genial idea that a generalization of the integral concept might come from dividing up the interval of variation of the dependent variable (instead of the independent variable's field as in Riemann's theory) the genius of Lebesgue has created a large and growing domain of analysis whose great importance cannot as yet be accurately estimated, but whose value in dealing with the more recondite problems of analysis is amply exemplified in these two volumes.

M. B. PORTER.

AUSTIN, TEXAS.

EXTRACT FROM A LETTER FROM PROFESSOR DE LA VALLÉE POUSSIN.

Cher monsieur et collègue,

Vous citez une objection de Schoenflies à propos de l'énoncé du théorème: Si dans un intervalle (a, b) deux fonctions $f(x)$ et $f_1(x)$ ont leurs nombres dérivés supérieurs à droite: 1° finis en chaque point sauf *peut-être* dans un ensemble E_1 et 2° égaux sauf *peut-être* dans un ensemble de mesure nulle, les deux fonctions ne diffèrent que par une constante à moins que E_1 ne contienne un ensemble parfait.

M. Schoenflies trouve cet énoncé contradictoire parce que deux fonctions qui ne diffèrent que par une constante ont la meme derivée *partout.*

Le mot *peut-être* que j'ai souligné dans l'énoncé a manifestement un sens subjectif et je refère à l'incertitude où nous pouvons être sur l'égalité ou la non égalité des nombres dérivés. L'objet du théorème est d'ailleurs précisément de lever cette incertitude. Voilà du moins ce que j'ai pensé.

L'objection de M. Schoenflies est d'ailleurs contestable en elle-même. Il n'est pas faux de dire que deux fonctions qui ne diffèrent que par une constante ont des derivées égales *sauf dans un ensemble de mesure nulle.* Car les derivées peuvent être infinies dans un ensemble de mesure nulle et on est en droit de dire que deux quantités infinies ne doivent pas être considérées comme égales. Je ne tiens d'ailleurs à cet argument que contre M. Schoenflies.

Vous faites remarquer encore que la généralisation du théorème de Scheeffer est moins grande qu'elle ne paraît parce que M. Young a démontré que E_1 est ou bien dénombrable ou bien a la puissance du continu. Donc, ajoutez-vous, le théorème vaut seulement si E_1 est dénombrable.

Pour que cette conclusion fût exacte, il faudrait démontrer que E_1 est ou bien dénombrable ou bien contient un ensemble parfait. Je crois bien que cela est vrai de tout ensemble mesurable (B) mais est-ce la meme chose que le théorème énoncé de M. Young?

Je vous signale enfin que l'ouvrage sur lequel vous faites rapport a été entièrement brûlé le 27 août dernier à Louvain, avec tout le magasin de mon éditeur, le 3e jour de l'incendie de cette ville par l'armée allemande.

Je vous prie, cher collègue, d'agréer l'expression de mes remerciments et l'assurance de mon entier dévouement,

C. DE LA VALLÉE POUSSIN.

CAMBRIDGE, MASS.,
May 11, 1915.

The objection to the formulation of Scheeffer's theorem referred to in Schoenflies' Bericht, volume 2, page 317, was directed at Scheeffer's statement of it. The reviewer was under the impression that even as stated above the *peut-être* might be objectively interpreted.

THE REVIEWER.

ENUMERATIVE GEOMETRY.

Lehrbuch der abzählenden Methoden der Geometrie. By H. G. ZEUTHEN. Leipzig, Teubner, 1914. xii + 394 pp. Price (cloth) 17 Marks.

IN the preface to his Lehrbuch Zeuthen expresses his gratitude to the publishers, "that the researches, which I have delighted to pursue from youth to an advanced age, may now appear in their full sequence." The mathematical world also has reason for hearty gratitude, not only to the Teubner firm, of whose family of publications this book is a very worthy member, but much more to Zeuthen himself, that he has produced a book summing up most carefully and elegantly both the chief results and the most fertile methods of enumerative geometry.

Zeuthen must have wished more than once in writing this work that a book were not essentially a one-dimensional configuration. The greater part of the book could have been displayed most satisfactorily in a plane with an axis of methods perpendicular to an axis of subjects. This arrangement being impossible, the author chose to make his work primarily a text on methods, and so to devote each chapter to a single method or group of methods. Within each chapter the results are grouped according to the configurations to which they apply, usually in the following order: plane curves, surfaces (in S_3), space curves, line configurations. The defect