A CERTAIN CLASS OF FUNCTIONS CONNECTED WITH FUCHSIAN GROUPS.

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1. CONSIDER a Fuchsian group Γ of linear substitutions

$$V_i \equiv z_i = \frac{\alpha_i z + \beta_i}{\gamma_i z + \delta_i} \qquad (i = 1, 2, 3, \cdots) \tag{1}$$
$$\alpha_i \delta_i - \beta_i \gamma_i = 1,$$

that transform the unit circle into itself, and for which the unit circle is a natural boundary. The index i for which z_i approaches a point of the boundary we denote by ∞, so that $\lim_{i=\infty} (z_i) = e^{i\phi}$, where ϕ may have any value from 0 to 2π. Let $z_0 = z$ represent identity. Denote by $R_0 = R$ the fundamental region in which z lies, and by R_1, R_2, $\cdots$ the regions resulting from R by the substitutions V_i $(i = 1, 2, 3, \cdots)$. Let e_i be the greatest "elongation" of the boundary of R_i, i. e., the maximum distance between two points of the boundary of R_i; then, according to a theorem due to Bricard,* it is possible to circumscribe a circle C_i to the region R_i, such that its radius does not need to be greater than at most $e_i/\sqrt{3}$.

For $i \neq \infty$, the area A_i of R_i, being that of a singly connected region bounded by circular arcs, is finite, so that for the ratio of the area of the circle C_i to that of the region R_i we have

$$1 < \frac{\pi e_i^2}{3A_i} < M \qquad (i = 1, 2, 3, \cdots), \tag{2}$$

where M is a positive finite quantity > 1. But it can be shown that this inequality also exists when $\lim (i) = \infty$, or $\lim_{=\infty} (z_i) = e^{i\phi}$. Hence from (2) we get

$$3\Sigma A_i < \Sigma \pi e_i^2 < 3M\Sigma A_i, \tag{3}$$

* "Théorèmes sur les courbes et les surfaces fermées," *Nouvelles Annales de Mathématique*, 4. ser., vol. 14, pp. 19–25 (January, 1914).

in which the sums are extended over the whole group Γ. As $\Sigma A_i = \pi$ is a finite quantity we find that the sum of the areas of all circles C_i, and consequently the sum of the squares of the radii of all these circles is finite.

2. Choose now within R any two points a and b and a variable point z, so that the area formed by the euclidean triangle $z_i a_i b_i$ lies entirely within C_i. Now

$$|z_i - a_i| \leqq e_i; \quad |z_i - b_i| \leqq e_i,$$

hence

$$|z_i - a_i| \cdot |z_i - b_i| \leqq e_i^2,$$

and

$$\sum_{i=0}^{\infty} |z_i - a_i| \cdot |z_i - b_i| \leqq \sum_{i=0}^{\infty} e_i^2. \tag{4}$$

But, according to (3), $\sum_{i=0}^{\infty} e_i^2$ is a finite quantity. The left side of (4) is therefore an absolutely convergent series, for all values of z within R. The condition for uniform convergence within the whole domain is evidently also satisfied, so that we can state

THEOREM I. *The series*

$$\sum_{i=0}^{\infty} (z_i - a_i)(z_i - b_i)$$

extended over a Fuchsian group Γ, *with the unit circle as a natural boundary and* z, a, b *lying within the fundamental region of* Γ, *is a uniformly convergent series, and defines an analytic function within* R *that vanishes for* $z = a$ *and* $z = b$ *and has no infinities within* R. *The result is still valid when* $z_b = z_a$, *so that*

$$\sum_{i=0}^{\infty} (z_i - a_i)^2$$

also defines such a function which at $z = a$ *has a zero of the second order.*

3. The theorem may immediately be generalized. Choose for z and a any two points within the unit circle (excluding the boundary). The straight line joining them is cut by a finite number of polygons R_i into the segments $l_1, l_2, l_3, \cdots, l_r$. Any substitution $V_\lambda \equiv \begin{pmatrix} \alpha_\lambda \beta_\lambda \\ \gamma_\lambda \delta_\lambda \end{pmatrix}$ of the group Γ transforms the

straight segment from z to a into an arc of a circle from z_λ to a_λ and the segments l_i into arcs $l_{i\lambda}$ intercepted by the corresponding polygons arising from the substitution V_λ. Every arc $l_{i\lambda}$ is subtended by a chord $s_{i\lambda} < l_{i\lambda} < e_{i\lambda}$, where $e_{i\lambda}$ denotes the elongation of the polygon (region) $R_{i\lambda}$. From this follows immediately that

$$f_\lambda = | z_\lambda - a_\lambda | < e_{1\lambda} + e_{2\lambda} + \cdots + e_{r\lambda},$$

and

$$(f_\lambda)^2 < (e_{1\lambda} + e_{2\lambda} + \cdots + e_{i\lambda} + \cdots + e_{k\lambda} + \cdots + e_{r\lambda})^2.$$

From the inequality

$$2e_{i\lambda}\, e_{k\lambda} < e_{i\lambda}{}^2 + e_{k\lambda}{}^2$$

we derive without difficulty

$$(5) \qquad 2 \sum_{\substack{i=1,\, k=1 \\ i \neq k}}^{r} e_{i\lambda} e_{k\lambda} < (r-1)(e_{1\lambda}{}^2 + e_{2\lambda}{}^2 + \cdots + e_{r\lambda}{}^2).$$

Now

$$\sum_{\lambda=0}^{\infty} (f_\lambda)^2 = \sum_{\lambda=0}^{\infty} (e_{1\lambda}{}^2 + e_{2\lambda}{}^2 + \cdots + e_{r\lambda}{}^2) + 2 \sum_{\lambda=0}^{\infty} \sum_{\substack{i=1,\, k=1 \\ 1 \neq k}}^{r} e_{i\lambda} e_{k\lambda};$$

hence, according to (5),

$$(6) \qquad \sum_{\lambda=0}^{\infty} (f_\lambda)^2 < r \sum_{\lambda=0}^{\infty} (e_{1\lambda}{}^2 + e_{2\lambda}{}^2 + \cdots + e_{r\lambda}{}^2).$$

But

$$\sum_{\lambda=0}^{\infty} e_{i\lambda}{}^2 = \sum_{\lambda=0}^{\infty} e_{k\lambda}{}^2,$$

so that (6) reduces to

$$(7) \qquad \sum_{\lambda=0}^{\infty} (f_\lambda)^2 < r^2 \sum_{\lambda=0}^{\infty} e_\lambda{}^2.$$

The right side of this inequality is a finite quantity, so that the series on the left side is absolutely convergent. Hence

THEOREM II. *The series*

$$\sum_{\lambda=0}^{\infty} (z_\lambda - a_\lambda)^2$$

extended over a Fuchsian group with the unit circle as a natural boundary, where z and a are any two points within the unit circle and not on the boundary, when a is fixed, is an absolutely and uniformly convergent series of z for all points within and not on the boundary, and represents an analytic function in the neighborhood of all such points. It has a zero of the second order for $z = a$, and has the unit circle as a natural boundary.

4. This theorem admits of a further generalization. Choose any three points z, z', a within and not on the unit circle, and write $f_\lambda = | z_\lambda - a_\lambda |$, $g_\lambda = | z_\lambda' - a_\lambda |$. Assuming $f \neq 0$ $g \neq 0$, it is possible to find a positive finite number M such that the ratio $g_\lambda/f_\lambda < M$, $\lambda = 1, 2, 3, \cdots$, also when z_λ approaches a point on the unit circle. We have therefore $g_\lambda < Mf_\lambda$, and

$$f_\lambda g_\lambda < M f_\lambda^2,$$

and consequently

$$\sum_{\lambda=0}^{\infty} f_\lambda g_\lambda < M \sum_{\lambda=0}^{\infty} f_\lambda^2. \tag{8}$$

As the right side of this inequality is absolutely convergent, it follows that

$$\sum_{\lambda=0}^{\infty} f_\lambda g_\lambda$$

is an absolutely convergent series, and that consequently

$$\sum_{\lambda=0}^{\infty} (z_\lambda - a_\lambda)(z_\lambda' - a_\lambda) \tag{9}$$

is absolutely and uniformly convergent, and, for a and z' constant, defines an analytic function of z for all points within and not on the boundary of the unit circle. It vanishes for $z = a$ and has the unit circle as a natural boundary. Nothing is lost in the convergency proof of (9) by assuming z and z' fixed and a as variable. Hence putting in (9) $z = a$, $z' = b$ and $a = z$ we may state

THEOREM III. *The series*

$$\sum_{\lambda=0}^{\infty} (z_\lambda - a_\lambda)(z_\lambda - b_\lambda),$$

where a and b are any two points within and not on the unit circle, is absolutely and uniformly convergent and represents an analytic *function of z within the unit circle, which is a natural boundary of the function. It has $z = a$ and $z = b$ as zeros.*

5. Making use of the proposition that for an analytic function $F(z)$ which within a certain region has the character of a rational function, such that for any point z_0 of this region $F(z_0)$ exists,

$$(10) \qquad \lim_{z=z' \doteq z_0} \left(\frac{F(z) - F(z')}{z - z'} \right) = F'(z_0)$$

we may extend theorem III to an even more general type of functions. Let $\mathfrak{R}(z)$ be a rational function of z which for $z = 0$ does not become infinite. Putting $(z_\lambda - a_\lambda)(z_\lambda - b_\lambda) = u_\lambda$, $(z_\lambda' - a_\lambda')(z_\lambda' - b_\lambda') = u_\lambda'$, where z', a', b' denote a set like z, a, b, then as z_λ ($\lambda = \infty$) approaches a definite point $e^{i\phi}$ on the boundary a_λ, b_λ, a_λ', b_λ', z_λ' will approach the same point, and u and u' will approach zero as a limit. Consequently

$$(11) \qquad \lim_{z_\lambda \doteq e^{i\phi}} \left\{ \frac{\mathfrak{R}(u_\lambda) - \mathfrak{R}(u_\lambda')}{u_\lambda - u_\lambda'} \right\} = \mathfrak{R}'(0)$$

is a finite quantity, and as $\Sigma(u - u')$ is absolutely and uniformly convergent, also $\sum_{\lambda=0}^{\infty} \{\mathfrak{R}(u) - \mathfrak{R}(u')\}$ will be absolutely and uniformly convergent within the unit circle, except for a finite number of values of u and u', which are poles of $\mathfrak{R}(u)$, and their congruents in the group Γ.* Hence, with the expressions for u, u' and $\mathfrak{R}$ defined as above, we may state

THEOREM IV. *When a, b, a', b', z' are fixed, so that no u_λ' is a pole of $\mathfrak{R}(z)$, then*

$$\sum_{\lambda=0}^{\infty} [\mathfrak{R}\{(z_\lambda - a_\lambda)(z_\lambda - b_\lambda)\} - \mathfrak{R}\{(z_\lambda' - a_\lambda')(z_\lambda' - b_\lambda')\}]$$

extended over the whole Fuchsian group represents an analytic function of z, which has the same poles as those of $\mathfrak{R}(u)$ and their congruents, and which has the unit circle as a natural boundary.

It appears that in general these functions are not automorphic in the ordinary sense.

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* For a statement of formula (10) and its applications to trigonometric and elliptic functions see Schottky: "Ueber die Funktionenklasse, die der Gleichung $F\left(\frac{\alpha x + \beta}{\gamma x + \delta}\right) = F(x)$ genügt"; *Crelle*, vol. 143 (1913), pp. 1–24.