$$(g - av)_p = \lim_{S \doteq p} \frac{1}{\sigma} \int_S \left[\frac{\partial}{\partial x}(a_{11}v) + \frac{\partial}{\partial y}(a_{12}v) - a_1 v \right] dy$$
$$- \left[\frac{\partial}{\partial x}(a_{21}v) + \frac{\partial}{\partial y}(a_{22}v) - a_2 v \right] dx,$$

then

$$(vf - ug)_p = \lim_{S \doteq p} \frac{1}{\sigma} \int_S \left[a_{11}\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} \right) + a_{12}\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y} \right) \right.$$
$$\left. + \frac{a_1 - b_1}{2} uv \right] dy - \left[a_{21}\left(v\frac{\partial u}{\partial x} - u\frac{\partial v}{\partial x} \right) \right.$$
$$\left. + a_{22}\left(v\frac{\partial u}{\partial y} - u\frac{\partial v}{\partial y} \right) + \frac{a_2 - b_2}{2} uv \right] dx,$$

where p is any point within S.

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CONVERGENCE OF THE SERIES $\sum_{i=0}^{\infty}\sum_{j=0}^{\infty} \frac{x^i y^j}{i - j\gamma}$ (γ IRRATIONAL).

BY PROFESSOR W. D. MACMILLAN.

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The method of proof which is here used depends upon the properties of continued fractions. Any irrational number γ can be expanded as a simple continued fraction

$$\gamma = a_1 + \frac{1}{a_2 +}\frac{1}{a_3 +}\frac{1}{a_4 +} \cdots.$$

Let p_n/q_n be the nth principal convergent,* and P/Q be any intermediate convergent lying between p_{n-2}/q_{n-2} and p_n/q_n. Then

$$\frac{p_{n-2}}{q_{n-2}} < \frac{P}{Q} < \frac{p_n}{q_n} < \gamma < \frac{p_{n+1}}{q_{n+1}} < \frac{p_{n-1}}{q_{n-1}}$$

if n is odd, and

* The notation used here agrees with that of Chrystal's Algebra, Vol. II, Chap. XXXII.

$$\frac{p_{n-2}}{q_{n-2}} > \frac{P}{Q} > \frac{p_n}{q_n} > \gamma > \frac{p_{n+1}}{q_{n+1}} > \frac{p_{n-1}}{q_{n-1}}$$

if n is even.

For n either even or odd

$$\left|\frac{P}{Q} - \gamma\right| > \left|\frac{P}{Q} - \frac{p_n}{q_n}\right| = \left|\frac{p_{n-2} + kp_{n-1}}{q_{n-2} + kq_{n-1}} - \frac{p_n}{q_n}\right|$$

$$1 \leqq k \leqq (a_n - 1),$$

from which is derived

$$\left|\frac{P}{Q} - \gamma\right| > \frac{a_n - k}{q_n Q}.$$

Since $q_n = q_{n-2} + a_n q_{n-1}$ and $Q = q_{n-2} + kq_{n-1}$, we have

$$q_n = Q + (a_n - k)q_{n-1} < Q + a_n q_{n-1} < (a_n + 1)Q.$$

Hence

$$\left|\frac{P}{Q} - \gamma\right| > \frac{a_n - k}{q_n Q} \geqq \frac{1}{q_n Q} > \frac{1}{Q^2(a_n + 2)},$$

and likewise

$$\left|\frac{p_n}{q_n} - \gamma\right| > \frac{1}{q_n(q_{n+1} + q_n)} > \frac{1}{q_n^2(a_{n+1} + 2)}.$$

We have then

$$(1) \qquad | P - Q\gamma | > \frac{1}{Q(a_n + 2)}$$

and

$$| p_{n-1} - q_{n-1}\gamma | > \frac{1}{q_{n-1}(a_n + 2)}.$$

Let us suppose now that $a_n + 2 < M$ for every n, an hypothesis which is certainly satisfied by every simple quadratic surd $(m \pm \sqrt{n})/l$, where l, m and n are integers. Then if P/Q is any convergent, principal or intermediate, it follows from (1) that

$$(2) \qquad | P - Q\gamma | > \frac{1}{MQ}.$$

We shall show now that for any two integers whatever, i and j,

$$(3) \qquad | i - j\gamma | > \frac{1}{Mj}.$$

Let P_1/Q_1 and P_2/Q_2 be two successive convergents (principal or intermediate) such that $Q_1 \leqq j \leqq Q_2$. From the general theory of continued fractions it is known that P_1/Q_1 and P_2/Q_2 are closer approximations to γ than any other rational fractions whose denominators are less than Q_2. Consequently

$$\left| \frac{i}{j} - \gamma \right| > \left| \frac{P_1}{Q_1} - \gamma \right| > \frac{1}{MQ_1^2},$$

and therefore, since $j > Q_1$,

$$(4) \qquad | i - j\gamma | > | P_1 - Q_1\gamma | > \frac{1}{MQ_1} > \frac{1}{Mj}.$$

Consider now the series

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{x^i y^j}{i - j\gamma} \qquad (i + j > 0),$$

where γ is an irrational number which, when expressed as a simple continued fraction, satisfies the condition that $a_n + 2 < M$ for every n. Then we will have $| i - j\gamma | > 1/(Mj)$ and consequently

$$\frac{1}{| i - j\gamma |} < Mj;$$

so that

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{x^i y^j}{i - j\gamma} = \sum_{i=1}^{\infty} \frac{x^i}{i} + \sum_{i=0}^{\infty} x^i \sum_{j=1}^{\infty} \frac{y^j}{i - j\gamma} \ll \sum_{i=1}^{\infty} x^i$$

$$+ M \sum_{i=1}^{\infty} x^i \sum_{j=1}^{\infty} j y^j \ll \frac{x}{1 - x} + \frac{M}{1 - x} \cdot \frac{y}{(1 - y)^2},$$

which converges if $| x | < 1$ and $| y | < 1$. Therefore the series

$$\sum_{i=0}^{\infty} \sum_{j=0}^{\infty} \frac{x^i y^j}{i - j\gamma} \qquad (i + j > 0)$$

converges if $| x |$ and $| y |$ are both less than unity, which is somewhat remarkable in that the denominators, which have no lower limit, impose no restriction upon the radii of convergence of the series.

The restrictions upon γ in the above discussion can be considerably reduced. It is seen from (1), if P/Q is any convergent (principal or intermediate), that

$$|P - Q\gamma| > \frac{1}{Q(a_n + 2)},$$

where a_n is the first partial quotient above Q. Let us suppose now that

$$(5) \quad a_n + 2 < M(q_{n-1} + 1)(q_{n-1} + 2)\cdots(q_{n-1} + S - 1),$$

where S is any positive integer independent of n. Then, since $Q > q_{n-1}$,

$$a_n + 2 < M(Q + 1)\cdots(Q + S - 1),$$

so that

$$|P - Q\gamma| > \frac{1}{MQ(Q + 1)\cdots(Q + S - 1)}.$$

Then, just as before, if i and j are any two integers such that $Q \leqq j \leqq Q_1$, we shall have

$$|i - j\gamma| > |P - Q\gamma| > \frac{1}{MQ(Q + 1)\cdots(Q + S - 1)}$$

$$> \frac{1}{Mj(j + 1)\cdots(j + S - 1)}$$

and also

$$\frac{1}{|i - j\gamma|} < Mj(j + 1)\cdots(j + S - 1).$$

If then γ satisfies these new conditions the series

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{x^i y^j}{i - j\gamma} \ll \frac{x}{1 - x} + M\sum_{i=0}^{\infty} x^i y \sum_{j=1}^{\infty} j(j+1)\cdots(j+S-1)y^{j-1}.$$

But since

$$\sum_{j=1}^{\infty} j(j + 1)\cdots(j + S - 1)y^{j-1} = \frac{d^S}{dy^S}\left(\frac{1}{1 - y}\right) = \frac{S!}{(1 - y)^{S+1}}$$

we have

$$\sum_{i=0}^{\infty}\sum_{j=0}^{\infty}\frac{x^i y^j}{i - j\gamma} \ll \frac{x}{1 - x} + \frac{yMS!}{(1 - x)(1 - y)^{S+1}}$$

and therefore convergent provided $|x|$ and $|y|$ are both less than unity.

Corollary.—If the series $f(x, y) = \sum_{i=0}^{\infty}\sum_{j=0}^{\infty} a_{ij}x^iy^j \quad (i + j > 0)$ converges for $|x| < 1/\xi$, $|y| < 1/\eta$, so that

$$f(x, y) \ll \frac{N}{(1-\xi x)(1-\eta y)} - N,$$

and if γ is an irrational number which satisfies (5), then the series

$$F(x, y) = \sum_{1=0}^{\infty}\sum_{j=0}^{\infty} \frac{a_{ij}}{i - j\gamma} x^iy^j \qquad (i + j > 0),$$

converges provided $|x| < 1/\xi$ and $|y| < 1/\eta$. Furthermore

$$F(x, y) \ll \frac{N}{1-\xi x}\left[\xi x + \frac{MS!}{(1-\xi x)(1-\eta y)^{S+1}}\right].$$

It will perhaps be interesting to note the character of the condition that $a_{n+1} + 2 < Mq_n(q_n + 1)\cdots(q_n + S - 1)$. Let us suppose that $a_n = n!$. It is found then $q_{n-1} = (n-1)!(n-2)!\cdots 2! + \cdots$. It is sufficient then to take $M = 1$, $S = 2$, in order to satisfy the condition. If we suppose that $a_n = 10^{10^{n-1}}$ we find that $q_{n-1} = 10^{10^{n-2}}\cdot 10^{10^{n-3}}\cdot \cdots 10^{10^0} + \cdots$, and it is sufficient to take $M = 10$, $S = 10$. If however we suppose that $a_n = 10^{n!}$ then $q_{n-1} = 10^{(n-1)!+(n-2)!+\cdots+2!} + \cdots$, and there do not exist an M and an S which satisfy the condition.

Application of these Series.—(*a*) The function

$$W = \sum_{m=0}^{\infty}\sum_{n=0}^{\infty}\frac{\lambda^m\mu^n}{m - nz}, \quad |\lambda| < 1, \quad |\mu| < 1,$$

where z is a complex variable, is a holomorphic function of z everywhere except in the neighborhood of the positive real axis, which is a line of essential singularities. Nevertheless the value of the function is finite for those real, positive irrational values of z which satisfy the above condition; furthermore the function is continuous across the real axis at any one of these points. To show the continuity, let $z = \gamma$ be such a point and let $z = \gamma + t\cos\alpha + it\sin\alpha$ be a straight line which crosses the real axis at γ making an angle α with the real axis. Then

$$W = \Sigma\Sigma \frac{\lambda^m \mu^n}{(m - n\gamma - nt\cos\alpha) - i(nt\sin\alpha)}$$

$$= \Sigma\Sigma \frac{[m - n\gamma - nt\cos\alpha] + i(nt\sin\alpha)}{[m - n\gamma - nt\cos\alpha]^2 + n^2t^2\sin^2\alpha}\lambda^m\mu^n.$$

If now we write $W = W_1 + iW_2$, and for brevity suppose λ and μ real, we have

$$W_1 = \Sigma\Sigma\lambda^m\mu^n \frac{m - n\gamma - nt\cos\alpha}{[m - n\gamma - nt\cos\alpha]^2 + n^2t^2\sin^2\alpha},$$

$$W_2 = \Sigma\Sigma\lambda^m\mu^n \frac{nt\sin\alpha}{[m - n\gamma - nt\cos\alpha]^2 + n^2t^2\sin^2\alpha}.$$

Consider now

$$\frac{nt\sin\alpha}{[m - n\gamma - nt\cos\alpha]^2 + n^2t^2\sin^2\alpha}.$$

As a function of the variable t this expression has a maximum or a minimum for $n^2t^2 = (m - n\gamma)^2$. It has a maximum equal to

$$\frac{1}{(m - n\gamma)[(1 + \cos\alpha)^2 + \sin^2\alpha]}$$

for $nt = (m - n\gamma)$, and a minimum equal to

$$\frac{-1}{(m - n\gamma)[(1 + \cos\alpha)^2 + \sin^2\alpha]}$$

for $nt = -(m - n\gamma)$. Consequently

$$W_2 \ll \frac{|\sin\alpha|}{[(1 - \cos\alpha)^2 + \sin^2\alpha]} \cdot \Sigma\Sigma \frac{\lambda^m\mu^n}{|m - n\gamma|},$$

which is absolutely convergent. Whence W_2, and in the same manner W_1, is absolutely and uniformly convergent for all real values of t. Consequently W is a continuous function of z all along this straight line.

(*b*) Consider the linear partial differential equation

$$x_1\frac{\partial\phi}{\partial x_1} - \gamma x_2\frac{\partial\phi}{\partial x_2} = p_1\phi + p_2,$$

where γ is a positive irrational number which satisfies the condition $a_{n+1} + 2 < Mq_n(q_n + 1) \cdots (q_n + S - 1)$, and $p_1 = \Sigma\Sigma a_{ij} x_1^i x_2^j$, $p_2 = \Sigma\Sigma b_{ij} x_1^i x_2^j$, are two convergent power series in x_1 and x_2.

We will take first the homogeneous equation

$$x_1 \frac{\partial \phi}{\partial x_1} - \gamma x_2 \frac{\partial \phi}{\partial x_2} = p_1 \phi,$$

and put $\psi = \log \phi$. Then

$$x_1 \frac{\partial \psi}{\partial x_1} - \gamma x_2 \frac{\partial \psi}{\partial x_2} = p_1.$$

The solution of this equation is

$$\psi = \Sigma\Sigma \frac{a_{ij}}{i - j\gamma} x_1^i x_2^j + \text{an arbitrary function of } (x_1^\gamma x_2),$$

and by the above corollary this series has the same region of validity as p_1 itself. It follows therefore that $\phi = e^\psi$ also is a convergent power series in x_1 and x_2, if the arbitrary function is taken equal to zero.

Returning now to the equation

$$x_1 \frac{\partial \phi}{\partial x_1} - \gamma x_2 \frac{\partial \phi}{\partial x_2} = p_1 \phi + p_2,$$

let us take $\phi = \omega e^\psi$, where e^ψ is the function already determined, and ω is an unknown function. We have then

$$x_1 \frac{\partial \omega}{\partial x_1} - \gamma x_2 \frac{\partial \omega}{\partial x_2} = p_2 e^{-\psi} = \Sigma\Sigma c_{ij} x_1^i x_2^j,$$

where $\Sigma\Sigma c_{ij} x_1^i x_2^j$ is the expansion of $p_2 e^{-\psi}$ and is therefore a convergent series. The solution of this equation is

$$\omega = \Sigma\Sigma \frac{c_{ij}}{i - j\gamma} x_1^i x_2^j + \text{an arbitrary function of } (x_1^\gamma x_2),$$

which likewise is a convergent series. Consequently $\phi = (A + \omega) e^\psi$, where A is an arbitrary function of $(x_1^\gamma x_2)$, is a solution of the differential equation.

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