If we were to pass a final judgment upon the book before us it would be this: It is in some respects an improvement upon its predecessors; it is by no means the best book that the present author could have written.

J. L. COOLIDGE.

CAMBRIDGE, MASS.,
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MINKOWSKI'S WORKS.

Gesammelte Abhandlungen von Hermann Minkowski. Herausgegeben von D. HILBERT. Leipzig, B. G. Teubner, 1911. Band 1, xxxi + 371 pp. Band 2, iv + 465 pp. Two portraits.

MINKOWSKI's work divides itself naturally, and his collected works are divided, into four parts: Theory of quadratic forms, 242 pages, Geometry of numbers, 230 pages, Geometry, 180 pages, Physics, 163 pages. In addition to this the volumes before us contain the author's address on Dirichlet, 15 pages, and Hilbert's commemorative address on Minkowski, 27 pages. This heartful and touching tribute of a life-long friend and fellow-worker is in reality also a critical review of Minkowski's great achievements in mathematical science, and it may be that the best thing for us to do in reviewing these volumes would be to follow the example of the reviewer in another Bulletin* and translate the chief portions of that address. The availability of the address in the original, where it should be read as a whole, and, in abstracts, in French makes repetition here seem really unnecessary.

We are accustomed to precocious exhibitions of genius in mathematicians, and we often cite the case of Galois, who died in his twenty-first year after accomplishing work of which the fundamental importance was not and perhaps could not be appreciated until a much later date. Minkowski in his eighteenth year submitted to the Paris Academy a memoir on quadratic forms with integral coefficients which fills 142 pages of his collected works and which received the Grand Prix des Sciences mathématiques. Measured in pages, one-sixth of Minkowski's work was written before he was 18. His work of the next ten years deals almost exclusively with quadratic forms.

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* *Bull. Sci. Math.*, France, vol. 36 (1912), p. 73.

In the early nineties, when about twenty-eight years old, Minkowski is found opening up his great work on the geometry of numbers, which may be said to occupy about a decade of his life. His book, Geometrie der Zahlen, of which the first part appeared in 1896 in the midst of this decade, was never finished. (The short second part published by Hilbert after the author's death was really merely the conclusion of the fifth section of the first part.) It is to the geometry of numbers with its fundamental concept of convex bodies that we may probably attribute the greatest brilliancy of Minkowski's brilliant career; and to it, also, we may look for his most permanent impressions upon mathematical thought. His work on geometry is a natural corollary of that on the geometry of numbers, and the major part of it in his collected works is the hundred page memoir, previously unpublished, on the theory of convex bodies and in particular on the foundation of the surface-concept as applied to them.

At the close of his life, Minkowski was working on physics. He had printed in the Encyclopedia the article on Capillarity, which is a model of exposition. He had printed his memoir on electromagnetic phenomena in moving bodies, had delivered his address at Cologne on Raum und Zeit, and was preparing a contribution on the deduction of the electromagnetic equations for moving bodies from the point of view of electron theory. This last contribution prepared by M. Born, more from his reminiscences of conversations with the author than from Minkowski's unintelligible notes, is included in the collected works.

Just how much praise the future will attribute to this work on electrodynamics we cannot estimate. The posthumous contribution seems to have had little influence; the preceding memoir is written in a somewhat clumsy notation, appears at times to be translating rather blindly known results into a new notation, and contains some errors.* But the Cologne address is a gem. In it is formulated with satisfactory simplicity the proposition that the laws of physics are fundamentally relations between certain vectors or other geometric elements (affected by coefficients) attached to four-dimensional loci and that the laws which we observe are relations between the projections of those vectors or other ele-

* Wilson and Lewis, "Relativity," *Proc. Amer. Acad.*, vol. 48, p. 495 (1912).

ments on our space-time system of reference. This idea was at the basis of the work in the memoir on electromagnetic phenomena, but needed the address to emancipate it.

There is one point in Minkowski's work which is of negligible importance for that work but which has attracted such comment that it is worth discussing for itself. When introducing his matrical calculations, the author appends a footnote:* "Mann könnte auch daran denken, statt des Cayleyschen Matrizenkalküls den Hamiltonschen Quaternionenkalkül heranzuziehen, doch erscheint mir der letztere für unsere Zwecke als zu eng und schwerfällig." †

Silberstein in published papers and in his book on relativity has shown conclusively that the analytical work on relativity can be carried out with extreme simplicity of notation by the use of quaternions. The most casual comparison of Silberstein's analysis with that of Minkowski will reveal very strikingly the neat quaternion and the clumsy matrix. The possibility of using quaternions lies in the fact that the Lorentz group is a group of rotations (imaginary or non-euclidean) in the four-dimensional manifold of x, y, z, t, and that, as Cayley showed, quaternions may be used to determine four-dimensional rotations by the formulas $q' = QqQ$, $T^2Q = 1$, where q denotes position.

The question, however, remains whether quaternions, though neat, are really appropriate. That they are not is indicated by Silberstein's admission that he had tried a whole year in vain a great variety of quaternion operations for relativistic purposes before discovering Cayley's proposition.‡ But there is a more fundamental reason: A four-dimensional vector is not a quaternion. A four-dimensional vector becomes a quaternion only after the choice of a second (reference) vector which is the scalar axis. Thus in four-dimensional analysis, just as in three dimensions, a quaternion involves two vectors, though in a very different way. The use of quaternions in geometric analysis in four dimensions involves, therefore, an extraneous element, just as the use of cartesian coordinates in geometric analysis in a plane involves elements extraneous to the geometric problem.

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* Vol. 2, p. 375.

† L. Silberstein, Relativity, viii+295 pp., Macmillan, 1914, states on page 150: " Minkowski himself despised Hamilton's calculus of quaternions as ' too narrow and clumsy for the purpose ' in question."

‡ *Phil. Mag.*, May, 1912, p. 790.

When using quaternions we have to be very careful to distinguish results which are intrinsically geometric from results which are relative to the direction of reals. This may be illustrated from the theory of complex numbers. The transformation $z' = az + b$, where a, b, z, z' are vectors in a plane (which become complex numbers after the choice of a real axis), is a transformation of similitude, no matter what direction be chosen as the axis of reals, but the transformation is not independent of that choice.

In relativity the time axis is accidental to a particular observer or group of observers and should be chosen *after* the fundamental work is done, not before. The analysis which is really appropriate to the theory of relativity as conceived by Minkowski is Grassmann's. Even a vector analysis (such as that used by Lewis and me, loc. cit.) assumes an origin, which is theoretically "de trop," though practically not much in the way. Is it not unfortunate that Minkowski should have followed the English Cayley, referred to the Scot-Irish Hamilton, and ignored the German Grassmann? Should not some Geheimer Regierungsrat among his colleagues have given him secret directions to avoid such an unpatriotic scientific mésalliance?

E. B. Wilson.

SHORTER NOTICES.

Histoire des Mathématiques. Par Ch. Bioche. Paris, Belin Frères, 1914. vi+93 pp. Price, 1 fr. 75 c.

It is one of the strange anomalies in the making of books that France, where the best work in the history of mathematics was done in the eighteenth and early nineteenth centuries, should have done so little in this line in recent years. Montucla, who wrote the first interesting general history of the subject; Delambre and De la Lande who were his worthy successors; Bossut, whose style maintained well the earlier traditions; Libri, writing in France although Italian by birth, and writing with the style of a novelist; Chasles, putting more mathematics into his work than his predecessors,—all these men contributed very notably to the appreciation of the historical development of the science, and set a high standard of style if not always of scholarship. But of late France has produced no general histories of mathematics worthy the name.