and making a set of five assumptions. It appears that the most general solution, when n is greater than 2, is $\varphi^{-1}f^r\varphi(x)$, where the integer r is prime to n. The case $n = 2$ is discussed separately and a simple algorism is given for reducing all differentiable functions of order 2 to a single type.

F. N. COLE,
Secretary.

THE LEGENDRE CONDITION FOR A MINIMUM OF A DOUBLE INTEGRAL WITH AN ISOPERIMETRIC CONDITION.

BY DR. CHARLES ALBERT FISCHER.

(Read before the American Mathematical Society, February 28, 1914.)

THE Legendre, or second necessary, condition for a minimum of a double integral, where there is no isoperimetric condition, has been derived by Kobb,* where the equations of the surfaces involved are in parametric form, and by Mason,† where x and y are the independent variables. The analogous condition for the isoperimetric problem has been proved to be sufficient to insure a permanent sign to the second variation,‡ but it has not been proved to be necessary.

In the present paper this condition,

$$h_{pp}(x, y, z, p, q; \lambda)h_{qq}(x, y, z, p, q; \lambda) - h_{pq}^2(x, y, z, p, q; \lambda) \geqq 0,$$

or expressed in parametric form,

$$H_{11}(x, y, z, x_u, x_v, \cdots, z_v; \lambda)H_{22}(x, y, z, x_u, x_v, \cdots, z_v; \lambda) - H_{12}^2(x, y, z, x_u, x_v, \cdots, z_v; \lambda) \geqq 0,$$

is proved to be necessary for either a maximum or a minimum.

Given two functions $f(x, y, z, p, q)$ and $g(x, y, z, p, q)$ and a surface

$$S: \qquad z = z(x, y)$$

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* "Sur les maxima et les minima des intégrales doubles," *Acta Mathematica*, vol. 16 (1892), p. 108.

† "A necessary condition for an extremum of a double integral," BULLETIN, vol. 13 (1907), p. 293.

‡ Kobb, *Acta Mathematica*, vol. 17 (1893), p. 331.

which satisfies the Lagrange differential equation

$$(1)\qquad h_z(x, y, z, p, q; \lambda) - \frac{\partial}{\partial x} h_p(x, y, z, p, q; \lambda) - \frac{\partial}{\partial y} h_q(x, y, z, p, q; \lambda) = 0;^*$$

it is desired to find a second condition which must be satisfied if the surface S gives a value to the integral

$$(2)\qquad J = \int\int_\Omega f(x, y, z, p, q)dxdy$$

as small as that given by any other admissible surface in the neighborhood of S. The region Ω is assumed to be bounded by a curve L of class D',† without double points, and the number of its intersections with any line parallel to either of the axes is assumed to be less than a fixed constant. The function $z(x, y)$ is assumed to be of class C'' in Ω, as are also the functions f and g in the neighborhood of S. A surface is said to be admissible if it is of class D', intersects S along a space curve which projects into L, and gives the same value as S to the double integral

$$K = \int\int_\Omega g(x, y, z, p, q)dxdy.$$

It will also be assumed that $z(x, y)$ is not a solution of the equation

$$(3)\qquad g_z(x, y, z, p, q) - \frac{\partial}{\partial x} g_p(x, y, z, p, q) - \frac{\partial}{\partial y} g_q(x, y, z, p, q) = 0.$$

If a one parameter family of admissible surfaces

$$\bar{S}: \qquad z = z(x, y) + \epsilon\bar{\zeta}(x, y, \epsilon)$$

is given, where the function $\bar{\zeta}(x, y, 0)$ and its partial derivative $\zeta_\epsilon(x, y, 0)$ are of class D', and this value substituted for z in the function f, the first variation of J must vanish because of

* Bolza: Vorlesungen über Variationsrechnung, p. 662.
† Bolza, loc. cit., p. 63.

equation (1). The second variation is found to be

$$(4)\qquad \delta^2 J = \epsilon^2 \int\int_\Omega (f_{zz}\bar{\zeta}^2 + 2f_{zp}\bar{\zeta}\bar{\zeta}_x + 2f_{zq}\bar{\zeta}\bar{\zeta}_y + f_{pp}\bar{\zeta}_x^2 + 2f_{pq}\bar{\zeta}_x\bar{\zeta}_y + f_{qq}\bar{\zeta}_y^2 + f_z\bar{\zeta}_\epsilon + f_p\bar{\zeta}_{x\epsilon} + f_q\bar{\zeta}_{y\epsilon})dxdy.$$

Since the surfaces considered are admissible, $\delta^2 K$, which can be evaluated in the same way, vanishes. If it is multipled by λ and added to $\delta^2 J$, and then Green's theorem* is applied, equation (4) becomes

$$\delta^2 J = \epsilon^2 \int\int_\Omega (h_{zz}\bar{\zeta}^2 + 2h_{zp}\bar{\zeta}\bar{\zeta}_x + \cdots + h_{qq}\bar{\zeta}_y^2)dxdy + \epsilon^2 \int\int_\Omega (h_z - \frac{\partial}{\partial x}h_p - \frac{\partial}{\partial y}h_q)\bar{\zeta}_\epsilon\, dxdy,$$

where as usual $h = f + \lambda g$. The last integral vanishes on account of equation (1), leaving

$$(5)\qquad \delta^2 J = \epsilon^2 \int\int_\Omega (h_{zz}\bar{\zeta}^2 + 2h_{zp}\bar{\zeta}\bar{\zeta}_x + \cdots + h_{qq}\bar{\zeta}_y^2)dxdy.$$

It will now be proved that if there is a point on S where the inequality

$$(6)\qquad h_{pp}h_{qq} - h_{pq}^2 < 0$$

is satisfied, the function $\bar{\zeta}(x, y, \epsilon)$ can be chosen in such a way that $\delta^2 J$ will be negative, and consequently there is no minimum.

If there is such a point there must be a region including the point where inequality (6) is satisfied. Two distinct points P_0 and P_0' in such a region can then be chosen, whose coordinates will be called x_0, y_0, $z(x_0, y_0)$ and x_0', y_0', $z(x_0', y_0')$. Since $z(x, y)$ is not a solution of equation (3), it can be assumed that

$$(7)\qquad g_z(x_0', y_0', \cdots) - \frac{\partial}{\partial x_0'} g_p(x_0', y_0', \cdots) - \frac{\partial}{\partial y_0'} g_q(x_0', y_0', \cdots) \neq 0.$$

* Bolza, loc. cit., p. 654.

The expression

$$h_{pp}(x_0, y_0, \cdots) \cos^2 \alpha + 2h_{pq}(x_0, y_0, \cdots) \cos \alpha \sin \alpha + h_{qq}(x_0, y_0, \cdots) \sin^2 \alpha$$

must vanish for two positive values of α less than π, and changes its sign when and only when α passes through one of them. It follows that a positive constant k^2 and a finite interval can be chosen in such a way that if α_1 and α_2 are any two angles in the interval, the inequalities

$$(8) \quad h_{pp} \cos^2 \alpha_i + 2h_{pq} \cos \alpha_i \sin \alpha_i + h_{qq} \sin^2 \alpha_i < - k^2 \qquad (i = 1, 2),$$

are satisfied at P_0, and since h_{pp}, h_{pq} and h_{qq} are continuous, they are satisfied at every point of S in a neighborhood of P_0. If there is given any positive constant δ it is possible to select two distinct angles α_1' and α_2' near to a root of the equation

$$h_{pp}(x_0', y_0', \cdots) \cos^2 \alpha + 2h_{pq}(x_0', y_0', \cdots) \cos \alpha \sin \alpha + h_{qq}(x_0', y_0', \cdots) \sin^2 \alpha = 0,$$

such that the inequalities

$$(9) \quad | h_{pp} \cos^2 \alpha_i' + 2h_{pq} \cos \alpha_i' \sin \alpha_i' + h_{qq} \sin^2 \alpha_i' | < \delta \qquad (i = 1, 2),$$

are satisfied at every point of S in a neighborhood of P_0'. For convenience these angles will be chosen so that $\alpha_2 - \alpha_1 = \alpha_2' - \alpha_1'$.

Two rhombuses R and R' will be defined as follows:* R is bounded by the lines

$$d - u_1 = 0, \quad d - u_2 = 0, \quad d + u_1 = 0, \quad d + u_2 = 0,$$

where

$$u_i = (x - x_0) \cos \alpha_i + (y - y_0) \sin \alpha_i, \quad (i = 1, 2),$$

and R' by the lines

$$d - u_1' = 0, \quad d - u_2' = 0, \quad d + u_1' = 0, \quad d + u_2' = 0,$$

where

$$u_i' = (x - x_0') \cos \alpha_i' + (y - y_0') \sin \alpha_i'.$$

* Compare with Mason, loc. cit., p. 295.

The constant d will be taken so small that R and R' do not overlap, and are entirely within the projections on the x, y-plane of the respective neighborhoods in which the inequalities (8) and (9) are satisfied. A function $\zeta(x, y)$ will be defined as identically zero outside of R, and inside of R as equal to the distance of the point x, y from the nearest side of R. That is,

$$\zeta(x, y) = d \mp u_i, \qquad (i = 1, 2),$$

where the sign and subscript of u are chosen so as to make ζ as small as possible. The function $\zeta'(x, y)$ will be chosen in the analogous way. Since $\alpha_2 - \alpha_1 = \alpha_2' - \alpha_1'$ the rhombuses R and R' are of the same size and shape and the equation

$$\int\int_R \zeta(x, y)dxdy = \int\int_{R'} \zeta'(x, y)dxdy$$

is satisfied.

The function $\bar{\zeta}(x, y, \epsilon)$ will now be defined by the equation

$$\epsilon\bar{\zeta}(x, y, \epsilon) = \epsilon\zeta(x, y) + \epsilon'(\epsilon)\zeta'(x, y),$$

where $\epsilon'(\epsilon)$ is to be determined by the condition that the surfaces $\bar{S}$ be admissible. The first variation of K is found to be equal to

$$\int\int_R \left(g_z - \frac{\partial}{\partial x} g_p - \frac{\partial}{\partial y} g_q\right) \zeta dxdy$$
$$+ \frac{d\epsilon'(0)}{d\epsilon} \int\int_{R'} \left(g_z - \frac{\partial}{\partial x} g_p - \frac{\partial}{\partial y} g_q\right) \zeta' dxdy = 0.$$

If the mean value theorem is applied to these integrals and the equation is solved for $d\epsilon'(0)/d\epsilon$, it becomes

$$\frac{d\epsilon'(0)}{d\epsilon} = - \frac{\left(g_z - \frac{\partial}{\partial x} g_p - \frac{\partial}{\partial y} g_q\right)_{x=\xi,\, y=\eta}}{\left(g_z - \frac{\partial}{\partial x} g_p - \frac{\partial}{\partial y} g_q\right)_{x=\xi',\, y=\eta'}},$$

where ξ, η is a point in R and ξ', η' a point in R'. The denominator cannot vanish if d is sufficiently small, because of inequality (7). Consequently a finite constant m can be chosen such that

$$\left|\frac{d\epsilon'(0)}{d\epsilon}\right| < m.$$

If the second variation of K is equated to zero, the coefficient of $d^2\epsilon'(0)/d\epsilon$ is also equal to

$$\int\int_{R'}\left(g_z - \frac{\partial}{\partial x}g_p - \frac{\partial}{\partial y}g_q\right)\zeta' dxdy \neq 0.$$

Consequently this second derivative exists. Therefore the functions

$$\bar{\zeta}(x, y, 0) = \zeta(x, y) + \frac{d\epsilon'(0)}{d\epsilon}\zeta'(x, y),$$
$$\zeta_\epsilon(x, y, 0) = \frac{d^2\epsilon'(0)}{d\epsilon^2}\zeta'(x, y)$$

exist and they have the required continuity.

When $\bar{\zeta}(x, y, \epsilon)$ is determined in this way equation (5) may be written

$$\delta^2 J = \epsilon^2\int\int_{R+R'}(h_{zz}\bar{\zeta}^2 + 2h_{zp}\bar{\zeta}\bar{\zeta}_x + 2h_{zq}\bar{\zeta}\bar{\zeta}_y)dxdy$$
$$+ \epsilon^2\int\int_{R}(h_{pp}\cos^2\alpha_i + 2h_{pq}\cos\alpha_i\sin\alpha_i + h_{qq}\sin^2\alpha_i)dxdy$$
$$+ \epsilon^2\left(\frac{d\epsilon'(0)}{d\epsilon}\right)^2\int\int_{R'}(h_{pp}\cos^2\alpha_i' + 2h_{pq}\cos\alpha_i'\sin\alpha_i' + h_{qq}\sin^2\alpha_i')dxdy,$$

where i takes the values 1 and 2 in the appropriate parts of R and R'. It can now be easily proved that

$$\delta^2 J < -\epsilon^2 A(k^2 - Md(d + 4)(1 + m^2) - \delta m^2),$$

where M is the largest of the maxima of the numerical values of h_{zz}, h_{zp} and h_{zq}, and A is the area of R.* Since d and δ can be taken as small as is desired without affecting k^2, they can be taken so small that $\delta^2 J$ will be negative and there is no minimum.

In a similar way it can be proved that there is no maximum if inequality (6) is satisfied, and the following theorem will be proved:

A necessary condition that the surface S furnish either a maximum or a minimum for the double integral (2), *relative to*

* Compare with Mason, loc. cit., pp. 295–6.

the admissible surfaces, is that

$$(10)\quad h_{pp}(x, y, z, p, q; \lambda)h_{qq}(x, y, z, p, q; \lambda) - h_{pq}^2(x, y, z, p, q; \lambda) \geqq 0,$$

at every point of S.

If the surface S is represented by the parametric equations

$$x = x(u, v), \quad y = y(u, v), \quad z = z(u, v),$$

the functions f and g must be replaced by $F(x, y\ z, x_u, x_v, \cdots z_v)$ and $G(x, y, z, x_u, x_v, \cdots, z_v)$ respectively. Then if the equations of $\bar{S}$ are written

$$x = x(u, v) + \epsilon\xi(u, v, \epsilon), \quad y = y(u, v) + \epsilon\eta(u, v, \epsilon),$$
$$z = z(u, v) + \epsilon\zeta(u, v, \epsilon),$$

the second variation becomes

$$(11)\quad \delta^2 J = \epsilon^2 \int\int_\Omega (H_{xx}\xi^2 + 2H_{xy}\xi\eta + \cdots + H_{z_vz_v}\zeta_v^2)dudv.$$

The values of J and K are assumed to be unchanged by any change in the parametric representation of S which leaves the surface S itself invariant. This furnishes a number of relations between the partial derivatives of F and G, among which are the following:*

$$(12)\quad \begin{aligned} &F_{x_ux_u} = F_{11}X^2, \quad F_{x_uy_u} = F_{12}XY, \quad F_{x_ux_v} = F_{12}X^2, \\ &F_{x_uy_v} + F_{x_vy_u} = 2F_{12}XY, \quad F_{x_vx_v} = F_{22}X^2, \quad F_{x_vy_v} = F_{22}XY, \end{aligned}$$

and the other formulas derived from these by permuting the letters x, y, z and X, Y, Z in the same way. The functions F_{11}, F_{12} and F_{22} are continuous and X, Y and Z are the direction cosines of the normal to S.

It will now be assumed that there is a function $\omega(u, v, \epsilon)$, such that

$$\xi(u, v, \epsilon) = \omega(u, v, \epsilon)X,$$
$$\eta(u, v, \epsilon) = \omega(u, v, \epsilon)Y,$$
$$\zeta(u, v, \epsilon) = \omega(u, v, \epsilon)Z.$$

If these values are substituted in the integrand of equation (11), it becomes a quadratic form in ω, ω_u, ω_v. The coefficient

* Kneser: Lehrbuch der Variationsrechnung, p. 282.

of ω_u^2 is seen to be

$$H_{x_ux_u}X^2 + 2H_{x_uy_u}XY + H_{y_uy_u}Y^2 + 2H_{x_uz_u}XZ + 2H_{y_uz_u}YZ + H_{z_uz_u}Z^2.$$

Equations (12), with F replaced by H, reduce this to the form

$$H_{11}(X^2 + Y^2 + Z^2)^2 = H_{11}.$$

Similarly the coefficients of $\omega_u\omega_v$ and ω_v^2 can be proved equal to $2H_{12}$ and H_{22} respectively. The other coefficients will be called H_{00}, $2H_{01}$ and $2H_{02}$ respectively, and equation (11) becomes

$$\delta^2J = \epsilon^2 \int\int_\Omega (H_{00}\omega^2 + 2H_{01}\omega\omega_u + 2H_{02}\omega\omega_v + H_{11}\omega_u^2 + 2H_{12}\omega_u\omega_v + H_{22}\omega_v^2)dudv.$$

This equation is in the same form as equation (5), and from this point on the argument is so nearly the same as in the non-parametric case that it need not be repeated here. The analogue of inequality (10) is seen to be

$$H_{11}(x, y, z, x_u, \cdots, z_v; \lambda)H_{22}(x, y, z, x_u, \cdots, z_v; \lambda) - H_{12}^2(x, y, z, x_u, \cdots, z_v; \lambda) \geqq 0.$$

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NOTE ON THE DERIVATIVE AND THE VARIATION OF A FUNCTION DEPENDING ON ALL THE VALUES OF ANOTHER FUNCTION.

BY PROFESSOR G. C. EVANS.

(Read before the American Mathematical Society, January 2, 1915.)

1. IN a recent article,* Fréchet has given a treatment of the differential of a function depending on a curve, by making use of and evaluating Riesz's expression of a linear relation in terms of a Stieltjes integral. According to Fréchet, if $\overset{b}{\underset{a}{F}}[\varphi]$ depends on all the values of $\varphi(x)$ between a and b, then

* M. Fréchet, "Sur la notion de différentielle d'une fonction de ligne," *Transactions of the American Mathematical Society*, vol. 15 (1914), pp. 135–161.