THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and seventy-fifth regular meeting of the Society was held in New York City on Saturday, February 27, 1915. The attendance at the two sessions included the following thirty-nine members:

Professor Joseph Bowden, Professor E. W. Brown, Dr. T. H. Brown, Professor B. H. Camp, Dr. Emily Coddington, Professor F. N. Cole, Dr. G. M. Conwell, Professor Elizabeth B. Cowley, Dr. Louise D. Cummings, Dr. H. B. Curtis, Professor L. P. Eisenhart, Dr. C. A. Fischer, Professor T. S. Fiske, Professor W. B. Fite, Professor W. H. Garrett, Professor O. E. Glenn, Professor C. C. Grove, Professor G. H. Hallett, Professor H. E. Hawkes, Dr. Dunham Jackson, Mr. S. A. Joffe, Professor Edward Kasner, Professor C. J. Keyser, Dr. J. K. Lamond, Mr. P. H. Linehan, Professor James Maclay, Dr. H. F. MacNeish, Dr. E. J. Miles, Mr. G. W. Mullins, Dr. G. A. Pfeiffer, Dr. H. W. Reddick, Professor R. G. D. Richardson, Mr. P. R. Rider, Dr. Caroline E. Seely, Professor L. P. Siceloff, Professor Edwin R. Smith, Professor Oswald Veblen, Mr. R. A. Wetzel, Miss E. C. Williams.

The President of the Society, Professor Ernest W. Brown, occupied the chair, being relieved at the afternoon session by Vice-President Oswald Veblen. The Council announced the election of the following persons to membership in the Society: Professor J. V. Balch, Bethany College; Professor E. J. Berg, Union College; Mr. Millar Brainard, Chicago, Ill.; Mr. L. C. Cox, Purdue University; Mr. C. H. Forsyth, University of Michigan; Dr. H. C. Gossard, University of Oklahoma; Mr. M. S. Knebelman, Lehigh University; Dr. W. V. Lovitt, Purdue University; Dr. L. C. Mathewson, Dartmouth College; Mr. A. L. Miller, University of Michigan; Dr. Bessie I. Miller, Johns Hopkins University; Mr. I. R. Pounder, University of Toronto; Mr. L. L. Steimley, Indiana University; Mr. Chid-Cheow Yen, Tangshan Engineering College. Three applications for membership in the Society were received.

From the early days of the Society the informal dinners held in connection with the meetings have been a most valuable supplement to the scientific programme, furnishing opportunities for a general exchange of views, conferences on questions of mathematics and of general policies, and renewing old and making new acquaintance. To promote this social side of the Society's activities, the Council has recently appointed a committee, consisting of Professors Fiske, Hawkes, and Kasner, to arrange a dinner for each New York meeting and by giving publicity to these occasions to induce a large attendance. Notice of the dinner is sent out with the programme of the meeting, and members expecting to attend are requested to fill out the accompanying card and return it to the Secretary, in order that adequate accommodations may be provided. The result at the recent annual meeting was the large attendance of seventy members. The February meeting is always, for several reasons, a smaller affair. Yet on this occasion nineteen members spent a very enjoyable evening together.

The following papers were read at this meeting:

(1) Professor M. FRÉCHET: "Sur les fonctionnelles bilinéaires."

(2) Professor A. S. HATHAWAY: "Gamma coefficients."

(3) Mr. P. H. LINEHAN: "Equilong invariants of irregular and regular analytic curves."

(4) Professor B. H. CAMP: "Multiple integrals over infinite fields."

(5) Mr. A. R. SCHWEITZER: "On the methods of mathematical discovery."

(6) Mr. P. R. RIDER: "An extension of Bliss's form of the problem of the calculus of variations, with applications to the generalization of angle."

(7) Professor E. B. WILSON: "The Ziwet-Field note on plane kinematics."

(8) Professor O. E. GLENN: "Ternary transvectant systems."

(9) Dr. E. J. MILES: "Note on the application of the calculus of variations to a problem in mechanics."

(10) Professor A. B. FRIZELL: "The permutations of the natural numbers cannot be well ordered."

(11) Mr. C. H. FORSYTH: "Osculatory interpolation formulas."

(12) Mr. J. F. RITT: "A function of a real variable with any desired derivatives at a point."

(13) Mr. J. F. RITT: "On Babbage's functional equation."

Professor Fréchet's paper was communicated to the Society through Professor D. R. Curtiss. Mr. Ritt was introduced by Professor Kasner. In the absence of the authors the papers of Professor Fréchet, Professor Hathaway, Mr. Schweitzer, Professor Wilson, Professor Frizell, and Mr. Forsyth were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Fréchet investigates operations $U_{f,\phi}$ which are linear with respect to f and to ϕ separately. Generalizing a known theorem of F. Riesz, he proves that $U_{f,\phi}$ may be represented as follows:

$$U_{f,\phi} = \int_a^{a'}\int_b^{b'} f(s)\phi(t)d_sd_tu(s, t),$$

the double integral on the right being defined as the limit of

$$\Sigma_i\Sigma_j f(\xi_i)\phi(\eta_j)\Delta_{i,j}u,$$

where

$$\Delta_{i,j}u = u(s_i, t_j) - u(s_i, t_{j-1}) - u(s_{i-1}, t_j) + u(s_{i-1}, t_{j-1}).$$

A striking result is that the function $u(s, t)$ need fulfill no other condition than that $|\Sigma_i\Sigma_j\epsilon_i\epsilon_j'\Delta_{i,j}u|$ have a finite upper bound whatever may be the signs of ϵ_i, ϵ_j', provided $|\epsilon_i| = |\epsilon_j'| = 1$. This representation by generalized double integrals is then applied to second differentials of continuous operations.

2. Professor Hathaway defines a gamma coefficient of coordinates $x, y, \cdots$ and parameters $a, b, \cdots$. The coefficient of two dimensions is

$$[axby] = (ax + by)\Gamma(x + y)/\Gamma(x + 1)\Gamma(y + 1),$$

and similarly for any dimension of coordinates.

This coefficient is shown to be constant upon an axis and, at any point, equal to the sum of its values at the points which precede it by a unit in the direction of each axis.

Taking integral values, none negative, for coordinates and parameters, an integral function is formed of corresponding arguments $p, q, \cdots$, consisting of the sum of all terms with coefficients of a given weight $n = ax + by + \cdots$, the coordinates of a coefficient being the exponents of the corresponding arguments. If m be the greatest parameter, it is shown that this function equals the sum of the nth powers of the roots of the equation

$$x^m = px^{m-a} + qx^{m-b} + \cdots.$$

This is a generalization of the formula for s_n given by Waring and proved by Serret (Cours d'Algèbre supérieure, volume I, page 445, article 196) from the formula of Lagrange. Waring's formula is deduced from the above by taking a, b, $\cdots$, m, as the natural numbers 1, 2, $\cdots$, m. When some parameters are equal, the general formula obtained above cannot be deduced from that of Waring. For example, the above result becomes the multinomial theorem for $a = b = \cdots = m = 1$; but Waring's formula, in the same case, is $(p + q + \cdots)^n = (p + q + \cdots)^n$.

Further properties of these coefficients will be the subject of a future paper.

3. Irregular analytic curves

$$v = \alpha_q u^{q/p} + \alpha_{q+1} u^{(q+1)/p} + \alpha_{q+2} u^{(q+2)/p} + \cdots$$

(u, v being Hessian line coordinates) are shown by Mr. Linehan to possess invariants under the group of equilong transformations of the plane except when $p = 2$. For each of the three cases which arise when invariants exist, the simplest invariant is derived.

The simplest invariant of a regular analytic curve under the linear equilong transformations of the plane is also obtained.

4. Professor Camp's immediate object in this paper is to prepare a foundation for a discussion, to be given later, of multiple and iterated integrals containing parameters, in which the integrations are extended over infinite fields. For this purpose it is necessary to coordinate the various definitions of these multiple integrals which have been given in recent years, to consider more fully than has been done heretofore the conditions of their existence, their relation to the iterated integrals, and certain of their fundamental properties.

5. In a previous article* Mr. Schweitzer stated a heuristic

* *Revue de Métaphysique et de Morale*, March, 1914.

principle of comparison based on identity, or principle of unification of terms. In the present paper is stated a principle of comparison antithetic to the preceding and based on diversity. The latter is called the principle of "furcation" of terms and is stated as follows:

The existence of dissimilarities between given terms implies the existence of dissimilar general terms which underlie or embrace the particular terms.

It is important to recognize the interdependence of the two principles. Thus by replacing "dissimilarities," "dissimilar" by "similarities" (resemblances), "similar" respectively, one obtains a possible interpretation of the author's principle of unification. The principle of "furcation" is useful in providing, directly or indirectly, problems of conflict for solution by means of the principle of unification.

An interesting special case of the principle of furcation is provided by equivalent mathematical theories or concepts which diverge under generalization. An instance of this kind is the author's "trifurcative" generalization* of the betweenness relation in the foundations of geometry. Many other illustrations might be given.

A concluding part of the author's paper is devoted to a critique of a remarkable article by J. T. Merz, History of European Thought of the Nineteenth Century, volume II, pages 627–740: "Development of mathematical thought." As a critical basis, the author's article on the working hypotheses of mathematics (cited above) is used.

6. Bliss has developed a theory of the calculus of variations for integrals of the form $I = \int f(x, y, \tau)ds$, where $\tan \tau$ is the slope of the curve considered. Mr. Rider's paper extends the theory to the integral $I = \int f(x, y, z, \tau, \sigma)ds$, where σ is the angle that a space curve makes with its own projection in the xy-plane, and τ is the angle that this projection makes with the x-axis. Necessary and sufficient conditions are studied.

A transversal surface is used in giving certain generalized definitions of angle and of solid angle. Incidentally, the differential equations of geodesics and generalized geodesics on this surface are derived. The definitions of angle reduce

* *American Journal*, vol. 31 (1909), p. 366.

under proper conditions to the definitions used in euclidean geometry.

7. Referring to the adoption by Ziwet and Field of the operators i and $e^{\phi i}$ of Burali-Forti and Marcolongo, Professor Wilson points out (1) that these operators are not algebraic as their form indicates, (2) that the operator $e^{\phi \mathbf{u} \wedge}$, which is patently non-algebraic, will not only serve the same purposes, but in addition lead to the derivation by A. C. Lunn of the Euler-Rodrigues form for rotations, and (3) that the algebraic operator $e^{\phi i}$, used in connection with the Gibbs bivector, will also accomplish the ends desired by Ziwet and Field. This note will appear in the *American Mathematical Monthly.*

8. The methods for the construction of fundamental systems of invariant formations of ternary quantics, known up to the present time, are tentative. The purpose of Professor Glenn's paper is, first, to deduce the simplest possible algorithm for the construction of such systems. Instead of deriving all invariants by means of one transvectant operation, four such operations are introduced, which may, however, be united under one major process of making cogredient substitutions in a double mixed polar. The main theorem of the paper is: If two systems of forms are finite and complete, the system derived therefrom by the above major transvectant process is finite and complete. A method of deriving the irreducible members of such a system is given. By the theorems of the paper the twenty invariant formations of two conics, for instance, may be readily written down.

9. The problem considered by Dr. Miles is: Given a chord of definite length and variable density; to find its form for uniform horizontal distribution of the mass if the center of gravity lies as low as posssible.

The density is assumed to be a function of x, say $r(x)$, and a problem in variations of the Lagrangian type results. The functions y and r are then determined and the curve is found to be a parabola—the solution given in text books on mechanics.

10. By methods developed in previous papers read before the Society (Lincoln, November, 1914; Chicago, December,

1914, and New York, January, 1915), Professor Frizell shows that the assumption that the set of permutations of the natural numbers can be well ordered leads, by a line of reasoning due to Cantor, to two sets of permutations which are both in one-to-one correspondence with the whole set of permutations of the natural numbers but cannot be put into one-to-one correspondence with each other.

11. As is well known, whenever successive intervals are to be interpolated, separate curves are used in the several intervals. In osculatory interpolation these separate curves are required to have the same slopes and curvatures at their points of intersection for purposes of smoother graduation. The appropriate formulas to be used in interpolating several values in each interval depend upon the fundamental interpolation curve upon which they are based, and formulas have been derived based upon Newton's and Everett's interpolation formulas. Mr. Forsyth derives the formulas based upon Stirling's and Bessel's formulas together with a series of corrections of fifth differences to be used in connection with the four formulas in lieu of the formulas in succeeding intervals. The paper will appear in the *Quarterly Publications* of the American Statistical Association.

12. There exists no analytic function having the sequence of numbers $a_1, a_2, \cdots, a_n$, as derivatives for $x = 0$, unless $|a_n/n!|^{1/n}$ stays bounded as n increases indefinitely.

Professor Kasner suggested the desirability of learning whether a non-analytic function can be found when no analytic function exists. Mr. Ritt has shown that the function

$$\sum_{n=1}^{n=\infty} \frac{a_n x^n}{n!}(1 - e^{-1/(b_n x^2)}),$$

where $1 < b_n > |a_n|$, has the desired sequence of derivatives.

13. About 1815, Charles Babbage attempted to find all functions of which the nth iterative is the independent variable itself. He observed that if $f(x)$ is such a function, $\varphi^{-1}f\varphi(x)$ will also be one, where $\varphi(x)$ is arbitrary; but he had no means of knowing how general his solution was.

Mr. Ritt has commenced a rigorous discussion of Babbage's functions, limiting himself to the case of the real variable, and making a set of five assumptions. It appears that the most general solution, when n is greater than 2, is $\varphi^{-1}f^r\varphi(x)$, where the integer r is prime to n. The case $n = 2$ is discussed separately and a simple algorism is given for reducing all differentiable functions of order 2 to a single type.

F. N. COLE,
Secretary.

THE LEGENDRE CONDITION FOR A MINIMUM OF A DOUBLE INTEGRAL WITH AN ISOPERIMETRIC CONDITION.

BY DR. CHARLES ALBERT FISCHER.

(Read before the American Mathematical Society, February 28, 1914.)

THE Legendre, or second necessary, condition for a minimum of a double integral, where there is no isoperimetric condition, has been derived by Kobb,* where the equations of the surfaces involved are in parametric form, and by Mason,† where x and y are the independent variables. The analogous condition for the isoperimetric problem has been proved to be sufficient to insure a permanent sign to the second variation,‡ but it has not been proved to be necessary.

In the present paper this condition,

$$h_{pp}(x, y, z, p, q; \lambda)h_{qq}(x, y, z, p, q; \lambda) - h_{pq}^2(x, y, z, p, q; \lambda) \geqq 0,$$

or expressed in parametric form,

$$H_{11}(x, y, z, x_u, x_v, \cdots, z_v; \lambda)H_{22}(x, y, z, x_u, x_v, \cdots, z_v; \lambda) - H_{12}^2(x, y, z, x_u, x_v, \cdots, z_v; \lambda) \geqq 0,$$

is proved to be necessary for either a maximum or a minimum.

Given two functions $f(x, y, z, p, q)$ and $g(x, y, z, p, q)$ and a surface

$$S: \qquad z = z(x, y)$$

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* "Sur les maxima et les minima des intégrales doubles," *Acta Mathematica*, vol. 16 (1892), p. 108.

† "A necessary condition for an extremum of a double integral," BULLETIN, vol. 13 (1907), p. 293.

‡ Kobb, *Acta Mathematica*, vol. 17 (1893), p. 331.