courses to provide both the medical and the engineering training which such officers require. But it is not expected that these courses should be a part of the training of every physician and of every engineer, nor even that every school of medicine or of engineering should establish such courses.

There already exist under university or government control several stations or institutions devoted to research in the problems of engineering. This research is naturally largely experimental, and these stations provide the opportunity of a career for the man who has the aptitude and the desire for this work. If such stations enter upon the field of the mathematical problems of engineering the door will be opened to a career in this line also.

With a definite demand for men competent to attack the mathematical problems of engineering will come the inducement for men to train themselves for the work, and with this inducement, a demand for suitable courses. To meet this demand a few institutions, already strong in both mathematics and engineering, may well organize graduate courses analogous to those which now lead to the degree of Doctor of Public Health.

FUNCTIONS OF LINES.

Leçons sur les Fonctions des Lignes. Par VITO VOLTERRA. Recueillies et rédigées par JOSEPH PÉRÈS. Paris, Gauthier-Villars, 1913. 8vo. vi+230 pp.

THESE lectures were delivered by Volterra at the Sorbonne during the months from January to March, 1912, and were later published as one of the Borel series of monographs on the theory of functions. It would be difficult to determine precisely the historical origin of functions of lines. Special cases of such functions, for example the ordinary definite integral or the integrals of the calculus of variations, have occupied a large share of the attention of mathematicians since the beginnings of the calculus itself. But the conscious formulation of the definition of a function of a line and its derivative, and the study of a general theory, belong to a recent period of investigation in which Volterra has been an earliest pioneer. The development of our knowledge of functions of lines and their applications, since Volterra's first published papers on the subject in 1887, has been due largely to his enthusiasm and to his confidence in the future importance of the theory.

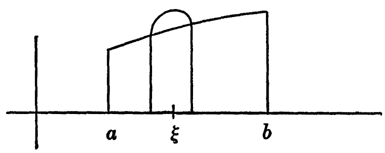
A function of a line may be conceived as a generalization from a function $F(y_1, y_2, \cdots, y_n)$ of a finite number of variables to a function

$$F|[f_a^b(x)]| \tag{1}$$

depending upon the infinity of y-values belonging to a curve $y = f(x)$ over an interval $a \leqq x \leqq b$. This process of passing from a conception involving a finite number to a conception involving an infinite number of mathematical symbols or operations is one of the most striking features of the mathematical thought of the present time. It is a process whose numerous possibilities of application Volterra repeatedly emphasizes in his Leçons, and which has been a favorite instrument in the prosecution of his varied researches.

Perhaps the most familiar example of such a process is the definite integral, the limit of a sum of a finite number of terms as the number of terms is indefinitely increased and the magnitude of each of the terms suitably diminished. Many centuries elapsed while this notion was haltingly developed into the basis of the modern integral calculus, and two more were required before the full importance was realized of the generalization in a similar way of other processes than sums. Then within a period of twenty-five years came the theories of functions of lines, of linear integral equations as generalizations from a finite to an infinite number of linear equations, of integro-differential equations as limiting conceptions corresponding to finite systems of simultaneous partial differential equations, and of other generalizations of the same sort with which the reader is doubtless familiar. It is with the three conceptions just mentioned specifically that the Lecons of Volterra are primarily concerned.

In an introductory chapter of great interest he has sketched the historical development of the fundamental principles of the integral calculus, and has shown how inevitably the tendencies of modern analysis and mechanics, but especially the mechanics of heredity, have led to the study of functions of lines. The theory of these functions is as yet in its infancy, and no final classification or complete discussion of its various branches can at present be made. But Volterra indicates briefly at this point the directions of the researches which have hitherto been made, and some of the questions which must be answered in the future.



In Chapters II, III, and IV the fundamental properties of functions of lines are developed. Let ΔF be the difference between the value of the function (1) taken along the curve $y = f(x) + \varphi(x)$ with the hump shown in the figure, and the value of F along the curve $y = f(x)$ without the hump; and let σ be the area of the hump. Then Volterra defines the derivative of F at the value ξ to be the limit

$$(2) \qquad F'|[f_a^b(x), \xi]| = \lim_{h=0,\ \epsilon=0} \frac{\Delta F}{\sigma},$$

where h is the length of the interval over which $\varphi(x)$ is different from zero, and ϵ is the maximum of the absolute value of φ. If F satisfies suitable restrictions this derivative will exist, and for a family of variations $y = f(x, \alpha)$ containing the curve $y = f(x)$ for $\alpha = 0$, the formula

$$\delta F = \int_a^b F'|[f(x), \xi]|\delta f(\xi, 0)d\xi,$$

so important in the calculus of variations, can be proved. The differentials in this formula are taken with respect to α. If F has also higher derivatives of all orders

$$F^{(n)}|[f_a^b(x), \xi_1, \xi_2, \cdots, \xi_n]|,$$

whose definitions will be readily inferred, and satisfies other suitable conditions, it will have an expansion of the form

$$F|[f(x) + \psi(x)]| = F|[f(x)]| + \int_a^b F'|[f(x), \xi_1]|\psi(\xi_1)d\xi_1$$
$$+ \frac{1}{2}\int_a^b \int_a^b F''|[f(x), \xi_1, \xi_2]|\psi(\xi_1)\psi(\xi_2)d\xi_1 d\xi_2 + \cdots,$$

a generalization of Taylor's formula.

It is not always true that two arbitrarily chosen functions of x and y will be the first partial derivatives of a third such function. The derivative (2) is really the partial derivative of F with respect to only one of the infinitely many variables which it involves, namely the value of y corresponding to $x = \xi$. It is not to be expected, therefore, that an arbitrarily chosen function (2) will always be the derivative of a function (1). Volterra deduces the conditions under which a function of the form (2) will be the derivative of a line function, and shows how the anti-derivative may be calculated with the aid of a generalization of Stokes' theorem.

If a curve $y = f(x)$ minimizes a line function F, the derivative (2) of F must vanish identically along the curve. For the line functions of the calculus of variations this condition leads to the usual ordinary differential equations due to Euler, but, as Volterra shows, it is easy to give an example for which the equation so found is of the type of a linear integral equation, or an integro-differential equation. The state of the theory at present does not justify a complete classification of all of the types which may arise.

Another interesting type of equations is met with when one attempts to generalize for double integrals the Jacobi-Hamilton theory associated with an integral of the form

$$F = \int_{x_0}^{x} V(x, y_1, \cdots, y_n, y_1', \cdots, y_n')dx.$$

The minimizing curves for such an integral are a family containing $2n$ parameters, and the parameters may in general be determined so that the resulting curve passes through two arbitrarily selected points, $(x_0, y_1^0, \cdots, y_n^0)$ and $(x, y_1, \cdots, y_n)$. The integral F is then a function of the coordinates of these points, called the extremal integral. It satisfies a partial differential equation of the form

$$\frac{\partial F}{\partial x} = H\left(x, y_1, \cdots, y_n, \frac{\partial F}{\partial y_1}, \cdots, \frac{\partial F}{\partial y_n}\right). \tag{3}$$

For a double integral

$$\int\int V\left(x, y, f, \frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}\right) dxdy$$

the minimizing surfaces are solutions $z = f(x, y)$ of a partial differential equation of the second order, and are determined by initial values $f_0(s)$ assigned arbitrarily to f on a closed contour L with length of arc denoted by s. The extremal integral is then a function $F|[f_0, L]|$ of the contour L and the values f_0, and the equation corresponding to (3) has the form

$$F'_L|[f_0, L, s]| = \frac{1}{2}\left(\frac{\partial f}{\partial s}\right)^2 - \frac{1}{2}F'^2_{f_0}|[f_0, L, s]|.$$

Equations of this type have been studied quite recently by Hadamard and Lévy, and are called by Volterra "équations aux dérivées fonctionelles."

The problem of determining the minimizing curves for a function of a line suggests the generalizations of the theory of implicit functions which are studied in Chapter IV. The equations

$$F_i(f_1, f_2, \cdots, f_n; \varphi_1, \varphi_2, \cdots, \varphi_m) = 0 \quad (i = 1, 2, \cdots, n),$$

which determine the variables f in terms of the φ's, correspond to a single equation of the form

$$(4) \qquad F|[f_a^b(x), \varphi_{a'}^{b'}(x), \xi]| = 0$$

when indices ranging over discrete integers are replaced by the index x ranging over the two continua $a \leqq x \leqq b$, $a' \leqq x \leqq b'$. The forms of this equation studied in detail by Volterra are, first, one corresponding to the linear equations which give rise to the theory of linear integral equations with variable upper limits; second, the case when the function F in (4) is the sum of two parts, one involving f and the other φ, and each part expansible by the generalization of Taylor's formula described above; and, finally, an equation (4) whose first differential has a special form. There remain, apparently, many cases yet to be studied, but the methods used by the author and the results which he has found are most interesting and suggestive.

In Chapter V Volterra begins his study of integro-differential equations with a special case. The equation considered is

$$(5) \qquad \Delta^2 u(t) + \int_0^t \left[\frac{\partial^2 u(\tau)}{\partial x^2} f(t, \tau) + \frac{\partial^2 u(\tau)}{\partial y^2}\varphi(t, \tau) + \frac{\partial^2 u(\tau)}{\partial z^2}\psi(t, \tau)\right] d\tau = 0,$$

where

$$\Delta^2 u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2}$$

and $u(x, y, z, t)$ is to be determined, an equation which he designates as the simplest integro-differential equation and the most important from the point of view of physics. The equation may be regarded as a generalization from a finite system

$$\Delta^2 u_i + \sum_{j=1}^{i-1} \left[a_{ij} \frac{\partial^2 u_j}{\partial x^2} + b_{ij} \frac{\partial^2 u_j}{\partial y^2} + c_{ij} \frac{\partial^2 u_j}{\partial z^2} \right] = 0 \quad (i = 1, 2, \cdots, n)$$

when the indices i and j are replaced by continuous variables t and τ, and the sum by the corresponding definite integral. The methods of treating this equation and the results found are analogous to those of the potential theory. There is an equation adjoint to (5) whose solutions satisfy a law of reciprocity with those of (5). There is a fundamental solution corresponding to the quotient $1/r$ in the potential theory, which Volterra determines with the help of the process of passing from a finite to an infinite system, so often mentioned above. Finally, formulas analogous to those of Green are deduced, expressing the values of a solution for $t = \theta$ at an arbitrarily selected point in the interior of an xyz-domain, in terms of its values for $0 \leqq t \leqq \theta$ on the boundary of the domain.

Chapters VI–VIII are devoted to the theory of elasticity and to concepts concerning functions of lines which have applications in this theory. The displacements in the interior of an elastic body are determined, in the classical theory of elasticity, as solutions of a system of partial differential equations, with initial conditions on the bounding surface defined by tensions or displacements there prescribed. This is for the case when displacements at a given time are supposed to depend only upon the tensions which exist at the same instant. For the hereditary theory of elasticity, when the displacements at a time t are supposed to depend upon the tensions at all times preceding t, Volterra shows that the former may be expressed in terms of the latter by means of line functions, and that the differential equations for the displacements then become integro-differential equations. For the case when the body is isotropic and the line functions mentioned above linear, an assumption analogous to the customary neglect of terms of an analytic function of degree higher than the first, Volterra sketches the theory with some detail. Theoretically the elastic problem so formulated involving heredity is completely solvable when displacements on the contour are given. The steps in the theory are analogous to those outlined for the integro-differential equation of Chapter V.

In Chapter VII a principle which Volterra calls "la condition du cycle fermé" is developed at length, and the importance of its applications in the theory of heredity is emphasized. If the condition of the closed cycle is satisfied in a problem involving heredity, the so-called coefficients of heredity will be functions permutable with unity in the sense described in Chapter IX, where the theory of such functions is systematically treated. The remainder of Chapter VII is devoted to a discussion of heredity in electromagnetic theory, and an application in this connection of the integro-differential equation of Chapter V.

In Chapter VIII a solution is given of the problem of the isotropic elastic sphere under the influence of heredity, when displacements on the contour are prescribed. Many of the principles involved in this special though very important problem are illustrations of the more general developments of later chapters.

The problem of the isotropic elastic sphere under the influence of heredity presents much greater difficulty when the tensions instead of the displacements on the surface of the sphere are the data given in advance. In order to attain a solution Volterra introduces a theory called the composition and permutability of functions, and he is led thereby to results which far surpass in generality the needs of this special problem. There are two kinds of composition, that of the first kind with its applications being developed in Chapters IX–XI. For two functions F_1 and F_2 the result of a composition of the first kind is a new function

$$\overset{*}{F}_1\overset{*}{F}_2(x, y) = \int_x^y F_1(x, \xi)F_2(\xi, y)d\xi.$$

The two functions are said to be permutable if

$$\overset{*}{F}_1\overset{*}{F}_2 = \overset{*}{F}_2\overset{*}{F}_1.$$

For a given set of functions the operation of composition applied to elements of the set is always associative and, if every pair is permutable, commutative. If the set so found is extended to contain all linear combinations of its functions of the form

$$c_0 + c_1F_1 + \cdots + c_nF_n$$

the properties of associativity and permutability of composition in the set are still preserved, provided that composition for two functions $c_1 + F_1$, $c_2 + F_2$ is defined by the equation

$$(c_1 + F_1)(c_2 + F_2) = c_1c_2 + c_2F_1 + c_1F_2 + \overset{*}{F}_1\overset{*}{F}_2.$$

But the set permits also a further important extension without disturbing these properties. For if in a convergent series the variables $z_1, z_2, \cdots, z_n$ are replaced by functions $F_1, F_2, \cdots, F_n$ of the set, and multiplications replaced by compositions, the new series will be convergent provided only that the moduli of $F_1, F_2, \cdots, F_n$ are finite. Furthermore the series function will be permutable with the ones previously defined.

By means of these sets of permutable functions, whose compositions obey the laws of algebra described above, a large variety of integral and integro-differential equations may be solved. For if the equation $F(z_1, z_2, \cdots, z_n) = 0$, where F is analytic at the origin, has a solution $z_n = f(z_1, z_2, \cdots, z_{n-1})$, the corresponding integral equation

$$F(\overset{*}{F}_1, \overset{*}{F}_2, \cdots, \overset{*}{F}_n) = 0,$$

formed by replacing the variables z by permutable functions as described above, has the solution

$$F_n = f(\overset{*}{F}_1, \overset{*}{F}_2, \cdots, \overset{*}{F}_{n-1}).$$

Furthermore an algebraic differential equation

$$\Phi\left(z_1, z_2, \cdots, z_n, F, \frac{\partial F}{\partial z_1}, \frac{\partial F}{\partial z_2}, \cdots\right) = 0$$

with a solution $F(z_1, z_2, \cdots, z_n)$ analytic at the origin, takes the form

$$\Psi\left(z_1, \cdots, z_n \mid \xi_0, \xi_1, \cdots, \xi_n \mid f, \frac{\partial f}{\partial z_1}, \cdots\right) = 0$$

when z_i is replaced by $\xi_i z_i$ $(i = 1, 2, \cdots, n)$, and F by f/ξ_0. If $\xi_0, \xi_1, \cdots, \xi_n$ are in turn replaced by permutable functions, and multiplications involving these quantities or f and its derivatives by compositions, then the last equation becomes an integro-differential equation

$$\psi\left(z_1, \cdots, z_n \mid \overset{*}{F}_0, \cdots, \overset{*}{F}_n \mid \overset{*}{f}, \frac{\overset{*}{\partial f}}{\partial z_1}, \cdots\right) = 0$$

whose solution is

$$f = \overset{*}{F}_0 \overset{*}{F}(z_1 \overset{*}{F}_1, \cdots, z_n \overset{*}{F}_n).$$

The function so defined has the remarkable property that it is convergent for all values of the variables z. Volterra shows in particular that the solutions of the integral or integro-differential equations considered in the preceding chapters can be reduced in this way to the solutions of ordinary algebraic or differential equations, the solutions of the latter being in many cases already known.

An interesting example is the equation

$$\frac{dU}{dz} = U + 1$$

with the known solution

$$U = z + \frac{z^2}{2!} + \cdots.$$

After substitutions of ξz for z, and V/ξ_0 for U, the differential equation becomes

$$\frac{dV}{dz} = \xi V + \xi \xi_0.$$

If ξ_0 is replaced by unity, and ξ by a function F, as agreed above, the new integral equation and its solution are

$$\frac{dV(z \mid x, y)}{dz} = F(x, y) + \int_x^y F(x, \xi) V(z, \xi, \eta) d\xi,$$

$$V(z \mid x, y) = zF(x, y) + \frac{z^2}{2!} \overset{*}{F}{}^2(x, y) + \cdots.$$

Further the addition formula

$$U(z + u) = U(z) + U(u) + U(z)U(u)$$

gives rise to an integral addition formula

$$V(z + u \mid x, y) = V(z \mid x, y) + V(u \mid x, y) + \int_x^y V(z \mid x, \xi)V(u \mid \xi, y)d\xi$$

for the solution of the integral equation. The integral transcendental function $V(z \mid x, y)$ so defined plays an important rôle in the solution of the problem of the elastic sphere with displacements on the surface given. The solution of the problem when tensions are given may also be effected with the help of the theory of permutable functions

The above results form the content of Chapters IX and X. In Chapter XI further questions concerning permutable functions of the first kind are discussed. Among them may be mentioned a definition of order of a function, the determination of all functions permutable with a given function of a given order, and the application of these results to the solution of integral equations at critical points.

In Chapters XII and XIII composition and permutability of the second kind are defined and applied to the solution of integral and integro-differential equations with fixed limits. The result of a composition of the second kind on two functions F_1 and F_2 is defined by the equation

$$\overset{**}{F_1}\overset{**}{F_2}(x, y) = \int_0^1 F_1(x, \xi)F_2(\xi, y)d\xi.$$

While the fundamental properties of such compositions are similar in many respects to those of composition of the first kind, there are still important differences, as one would expect from the analogy with the theories of integral equations with variable or fixed limits. It will perhaps be sufficient here to say that the underlying idea is again to pass from algebraic or differential equations to integral or integro-differential equations, but this time with limits fixed instead of variable. The solutions of the former types of problems will go into those of the latter by the substitution of compositions for multiplications.

The final chapter of the Lecons is devoted to a historical summary of the more recent developments in mathematical theories of mechanics, and a discussion of the place which the theory of heredity should take in this domain. Some writers have questioned the importance of discussions of heredity on the ground that the behavior of an elastic body, for example, is completely determined after a time t_0 by its state at $t = t_0$. According to this view the discrepancies between theory and observation are due to inadequate experimental facilities for the determination of interior initial conditions. Volterra, on the other hand, contends that just as bodies react upon each other at a distance, so it is possible that tensions and displacements separated by an interval of time may be related. Even if the contention of those who question the philosophical basis of the theory is granted, nevertheless the agreement between experimental results and the elastic theory involving heredity is in itself a justification. It seems clear to Volterra that, in view of the difficulty of determining initial conditions experimentally, and in the absence of other theories agreeing with experiment, the theory of heredity offers the only explanation at present possible of a large class of phenomena. He cites the experiments of Webster and Porter in the theory of sound as a cogent illustration, and his own success in the mathematical solution of problems in the theory of elasticity with heredity must be regarded as a potent argument in favor of his point of view.

G. A. BLISS.

SHORTER NOTICES.

List of Prime Numbers from 1 *to* 10,006,721. By D. N. LEHMER. Washington, D. C., Carnegie Institution of Washington, 1914. xvi+133 pp.

THERE are several reasons why number theorists will welcome most heartily the publication of this volume.

First, it answers with utmost directness the question, arising at almost every stage of number theoretic computation, whether or not a proposed number (under ten millions) is prime. Here the question of absolute accuracy of a table is paramount; the user of such a table has no practical means of checking the accuracy of an entry and if he relies upon an erroneous entry his conclusions will be wholly wrong. It is thus quite different from the case of ordinary tables (those of the values of a continuous function), since it is there only a question of approximation and a grossly erroneous error should be detected by the user of the table. The present table prob-