MATHEMATICAL METHODS IN PHYSICS.

Sur quelques Progrès récents de la Physique mathématique. Par VITO VOLTERRA, Clark University Lectures of 1909, published by Clark University, 1912. 82 pp.

Drei Vorlesungen über neuere Fortschritte der mathematischen Physik. Von VITO VOLTERRA, mit Zusätzen und Ergänzungen des Verfassers. Deutsch von Dr. ERNST LANELA. Sonderabdruck aus dem *Archiv der Mathematik und Physik,* III. Reihe, Band XXII, Heft 2/3. B. G. Teubner. Leipzig.

Lecons sur l'Intégration des Equations différentielles aux Dérivées partielles. Par VITO VOLTERRA. Professées à Stockholm. Nouveau tirage. Paris, Hermann. 1912. 3 + iv + 83 pp.

The first of these books consists of three lectures delivered at Clark University, and afterwards printed by the University.* They have since appeared in the second form in German in the *Archiv der Mathematik und Physik,* (3), 22 (1914), pages 97–182. In the latter form some of the details omitted in the original are supplied. The third book is a reprint of lectures delivered at Stockholm in 1906. There have been added some corrections, and some bibliographical notes. These lectures are striking examples of the intimate relationship between the advance of mathematics and that of physics.

The fundamental notion of the Stockholm lectures is that the theories of the propagation of heat, of hydrodynamics, elasticity, Newtonian forces, and electromagnetism can all be treated from a single point of view—reducing indeed to differential equations of the same general form but of three types, the facts and the processes used following the types. A good supplementary paper to read along with the first part of the lectures on differential equations, containing examples and more detail, is to be found in the *Annales de l'Ecole Normale,* (3), 24 (1907), page 411. The most interesting part of the lectures is the introduction of the notion, due to Professor Volterra, of *function of a line.* In the fifth lecture this notion appears, and is indeed the guide to a generalization of the

* The volume also contains lectures by Rutherford: "History of the alpha-rays from radio-active substances"; Wood: "The optical properties of metallic vapors"; Barus: "Physical properties of the iron carbides."

analytic functions of a complex variable. The particular function of a line used in this lecture is the line integral

$$V = u_0 + \int_A^B (Xdx + Ydy + Zdz),$$

where u_0 is a constant and the line integral extends from a point A to a point B. If we set $X' = (\partial Y/\partial z - \partial Z/\partial y)$, $Y' = (\partial Z/\partial x - \partial X/\partial z)$, $Z' = (\partial X/\partial y - \partial Y/\partial x)$, then, n being the outward normal to the surface enclosed, the line integral in question around a loop will, by Stokes's theorem, be the same as

$$W = \int\int (X' \cos nx + Y' \cos ny + Z' \cos nz)dA.$$

If we let the loop decrease and determine the limit of the ratio of W to the area enclosed, as the vanishing loop approaches a point, the limit in question is nothing else than the projection, on the normal to the surface at the point, of the vector whose components are X', Y', Z', that is, of the *curl* of the vector X, Y, Z. This projection, if found for a point on the path of integration, Professor Volterra calls the derivative of the function of the line V with respect to the surface, and he represents the curl by the symbols

$$X' = \partial V/\partial(yz) \quad Y' = \partial V/\partial(zx) \quad Z' = \partial V/\partial(xy).$$

If now there is a function u whose gradient is (X', Y', Z'), that is, if

$$\partial u/\partial x = X', \quad \partial u/\partial y = Y', \quad \partial u/\partial z = Z',$$

then the convergence of the gradient of u is zero, and u is harmonic since $\nabla^2 u = 0$. When these relations are satisfied we have u and V so related that one may be called the conjugate of the other. This is the generalization referred to of the theory of complex variables. An easy mathematical example is found by setting

$$u = x^2 + y^2 - 2z^2,$$

which is harmonic, and whose gradient is the vector

$$(2x, 2y, -4z).$$

It is conjugate to the function (integral around a loop)

$$V = \int (2zydx - 2zxdy),$$

since the curl of $(2zy, -2zx, 0)$ is $(2x, 2y, -4z)$, the gradient of u. An easy physical example is the field of potential at a fixed origin of a single magnet pole in empty space, as the pole is moved into all possible positions, which gives the function u; and the field of a circuit carrying a unit electric current, as it is moved into all possible positions and shapes, which gives the potential V at the fixed origin. The two functions are conjugate, the first a function of a point, the reciprocal of its distance from the origin; the latter a function of a line, the solid angle it subtends at the origin.

The function of a line of course need not be conjugate to a function u. In case X', Y', Z' is a vector whose convergence is zero, then it may be written as the curl of a vector X, Y, Z, the well-known relation of a vector to its vector potential; and the integral

$$\int\int (X' \cos nx + Y' \cos ny + Z' \cos nz) dA = \int (Xdx + Ydy + Zdz)$$

gives a function of a line. It need be remarked that these definitions of function of a line and the differential of a function of a line are generalized somewhat in the calculus of fonctionnelles.

The notion of monogenicity is extended, under the name isogenicity, to space of three dimensions in the following form. Two functions of a complex variable z are monogenic if their differentials at any point have a ratio f which is a function of the point but not of the direction of the differentials. Two functions of a line are isogenic if at each point of the line the vector derivative of each, that is, the curl of the vector which is to be integrated along the line, is parallel to the vector derivative of the other. In vector notation we would write the definition

$$V\nabla B_1 = fV\nabla B_2,$$

where f is independent of the derivative plane as defined above. The idea of function of a line may evidently be extended to functions of surfaces and hyperspaces, and leads into the calcul fonctionnel.

These functions of lines and surfaces, and for the most general case, of hyperspaces, enable one to understand the application of the methods of Jacobi and Hamilton to the problems of the calculus of variations. The simple integral of the ordinary theory may be considered as a function of its limits and of the values of unknown functions at the limits. The extensions to multiple integrals are easily suggested by this view of the procedure, namely, we must consider the integrals to be functions of the lines, or surfaces, or hyperspaces, that bound the space, and of the values of unknown functions on these contours.

The detailed treatment of the partial differential equations is to be found in the last four lectures, and reference to them is necessary to have a clear notion of them.

In the course at Clark University we find the dominant idea again to be the unifying principles of the application of mathematics to mechanics, elasticity, and mathematical physics. The first lecture is devoted to showing the reduction of physical problems to problems in the calculus of variations, the second to the advance in methods in elasticity, and the third to the problem of heredity, which leads to Volterra's integro-differential equations. These three lectures we will examine in some detail.

In the first there is given a reduction of the problem of electrodynamics into terms of the variation of the definite integral

$$P = \int dt \tfrac{1}{2} \iint \Sigma(\alpha_{rs} X_r X_s + \beta_{rs} L_r L_s) dA,$$

in which the quantities α, β are quite arbitrary. This variation in terms of purely arbitrary quantities shows us that we can devise an infinity of mechanical explanations or models of electrodynamic phenomena. But this analytic form of the problem enables us to introduce curvilinear coordinates, and thus consider curvilinear spaces. We also are enabled to find the integral invariants, and to apply Volterra's reciprocity theorem which corresponds to Green's theorem, and further to introduce the generalization of the Hamilton-Jacobi methods and the principle corresponding to stationary action and varying action. In order to accomplish this, the functions of lines, surfaces, and hyperspaces have to be used. We thus come into contact with the theory of the inversion of definite integrals, that is, the solution of linear integral equations, and with the study of functions of variables which run over assemblages of curves, of surfaces, etc. We are brought up to the functional calculus, or as it has been called, general analysis. The generalization of the Hamilton-Jacobi method replaces the canonical equations by partial derivatives, and the partial differential equation of Jacobi is replaced by a functional equation.

There is an intimate connection between the partial differential equation, its characteristics, and the theory of waves.* It is simply necessary to consider the variable t as on the same footing as the variables x, y, z. The bearing of this is shown in the closing considerations of the first lecture. For example, in the very simple case of the equation of a vibrating membrane

$$\partial^2 u/\partial t^2 - \partial^2 u/\partial x^2 - \partial^2 u/\partial y^2 = 0$$

we have the partial differential equation corresponding to the vanishing of the variation of the integral

$$V = \int\int\int [(\partial u/\partial t)^2 - (\partial u/\partial x)^2 - (\partial u/\partial y)^2] dxdydt.$$

The surfaces of discontinuous derivatives and variation of V always equal to zero are then the envelopes of the characteristic cones of the partial differential equation. The general question of waves is therefore considered, and Minkowski's universe noticed. This naturally leads to the Lorentz transformation, and to Poincaré's demonstration that under this transformation the integral whose variation was considered, and which may be called the action, remains invariant.

In the second lecture the development of methods in the theory of elasticity is taken up, particularly those that are connected with the ideas already mentioned. The two great classes of methods of integrating the differential equations of elasticity may be called the method of Green with its extensions, and the method of simple solutions. Green's method further has two divisions, in one of which the conception of Green alone is sufficient to solve the problem, in the other we must add consideration of the characteristics. Green's method starts with Laplace's equation, and depends upon a reciprocity theorem, by means of which from a fundamental solution one is enabled to determine a harmonic function inside a given region when its values on the contour are given. Betti carried the method of Green over into elasticity and extended the reciprocity theorem by the proposition: If two

* Encyclopédie des Math., II_4 1 (II, 22, 8).

systems of exterior forces determine two systems of displacements in an elastic body, the work done by either in producing the displacement due to the other is the same.

The second division of methods along the line started by Green is found in Kirchoff's work on the equation of retarded potential on four variables. He succeeded in solving it by means of a fundamental solution due to Euler, but in order to perceive the real difficulties in the way it is necessary to consider the problem on only three variables. The characteristics enter into the solution radically in this case. The lecturer shows the inherent difference between the cases of three dimensions and four dimensions. Again when the body is not simply connected and the functions can be polydromic the method of Green needs further extensions. The reciprocity theorem enters in as a theorem of the symmetry of a set of coefficients $E_{ij} = E_{ji}$, which occur in the linear equations that give the efforts in terms of the distortions.

The method of simple solutions has been given great power by the development of the theory of integral equations, and the determination of methods of expansion in series of orthogonal functions.

The third lecture introduces the new developments due to Professor Volterra himself and now well-known. These lead to the division of mechanics into the mechanics of no heredity, wherein the state of a system depends only upon the infinitesimally near states preceding, and the mechanics of heredity, in which the state depends upon all the preceding states, thus introducing an action at a time-distance. A simple example is used to illustrate the new problem, the dependence of the angle of torsion of a wire upon the moment of torsion. Instead of Hooke's law

$$\omega = KM,$$

we find it must be expressed by a more elaborate law dependent upon the time

$$\omega = KM(t) + \int_{t_0}^{t} M(t)\varphi(t, \tau)d\tau + \frac{1}{1\cdot 2}\int_{t_0}^{t} d\tau_1 \int_{t_0}^{t} d\tau_2 M(\tau_1)M(\tau_2)\varphi(t, \tau_1, \tau_2) + \cdots.$$

In the case of linear heredity this expression terminates with the second term. There arise now several questions.

(1) What is the significance of the coefficient $\varphi(t, \tau)$?

(2) When φ is known, how may M be found for a given ω?

(3) How is φ determined when it is unknown?

(4) Is it possible to extend these conceptions to the general problem of elasticity?

(5) Is it possible to extend these conceptions to the phenomena of magnetism and electricity?

(6) What phenomena will be amenable to this method of treatment?

For the first three questions it is found that the methods of integral equations are sufficient to furnish the answer. However for the fourth a new type of equation is in evidence, the integro-differential equation. We meet this indeed in the problem of the wire itself when we study its oscillations. The integro-differential equation involves the partial derivatives of the unknown function as well as integrals containing the unknown function. The answer to the fifth question leads also to integro-differential equations for the electromagnetic equations.

To resolve equations of this form a further extension of the method of Green is necessary, and a new reciprocity theorem arises. Fundamental solutions may be found and from these arises the complete solution.

In applying algebra to natural phenomena we have the great advantage that we can postpone to the last moment the specification of the constants that enter the natural problem, and are thus in a much better position to consider the related hypotheses. So too in the functional calculus in all its forms, we are able to postpone the specification of the functions entering the problem until the last moment, and are thus able to avoid vicious hypotheses in the beginning. Without these developments many types of problem become impossible of solution. Further, advances in mathematical physics come from the subsumption under one law of many different classes of phenomena.

In the translation of the Clark University Lectures many details of the transformations in the first lecture are supplied, to the relief of the general reader. Further, the references have been made more precise and have been inserted on the pages as footnotes instead of being collected at the ends of the lectures. Some minor corrections have been made. The section headings have been inserted over the sections.

This edition of these very valuable lectures will be welcomed.

The whole field of functional calculus is a new territory but recently open for settlement, though an adventurous investigator occasionally explored small parts of it in the past century. The important extensions of mathematics have come from the problems of inversion, such as the Galois theory, theory of ideals, differential equations, integral equations, and now the calcul fonctionnel. These developments of Professor Volterra are of the highest importance mathematically aside from all of their physical interest, for the reason that they furnish a very practical path of entry into this new field and occupy a considerable part of the field itself. Fortunately we can follow them more in detail in the two recent courses of his lectures, Leçons sur les fonctions des lignes, and Lecons sur les équations intégrales.

James Byrnie Shaw.

SHORTER NOTICES.

Mysticism in Mathematics. By Hastings Berkeley. Henry Frowde, Oxford University Press, 1910. vi + 264 pp.

It may be well to begin by stating what Mr. Berkeley's thesis is not. The kind of mysticism which he thinks he detects in mathematics is not any of the kinds of mysticism that one encounters in the history and philosophy of religion. It is not contended that devotion to mathematics begets or tends to beget in the devotee a sense of an immediate and ineffable union or identification with deity. It is not argued that there is any essential likeness between Euclid and Timæus or between Gauss and Angelus Silesius. What Mr. Berkeley calls mysticism in mathematics is, he says, so called by him because no other name seems to him so appropriate. It appears to be impossible to state with mathematical precision what the thing is. Of course the author is not to be blamed for that. He succeeds in the difficult enterprise of making the matter as clear to the reader as it is to the writer, and that is all that can be reasonably expected.

As nearly as we have been able to make it out, the thesis may be broadly stated to be that, owing to a kind of reaction of language and especially of highly symbolic language upon