ON A GENERALIZATION OF A THEOREM OF DINI ON SEQUENCES OF CONTINUOUS FUNCTIONS.

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WE propose in this note to give a generalization of the following theorem of Dini*: "If a monotonic sequence of functions continuous on a closed interval converges to a continuous function, the convergence is uniform."

The double sequence analogue of this theorem proves to be of importance in our generalization. We embody it in the following

LEMMA. *If a double sequence a_{mn} is monotonic non-decreasing in m for every n, and if $L_m L_n a_{mn} = L_n L_m a_{mn}$, all the limits being supposed to exist, then $L_m a_{mn}$ and $L_n a_{mn}$ converge uniformly and the double limit $L_{mn} a_{mn}$ exists and is equal to the iterated limits.*

The proof of the uniformity of convergence of $L_m a_{mn}$ is part of Theorem I of the paper by the author in the *Annals of Mathematics*, series 2, volume 14, page 81. This uniformity has as a direct consequence the existence of the double limit equal to the iterated limits, which in turn implies the uniformity of $L_n a_{mn}$.

For the purposes of generalization, consider a class $\mathfrak{D}$ of elements unconditioned excepting for the existence within the class of some definition of limit, i. e., some means of determining whether a sequence of elements has a limit and what this limit is.† Then it is possible to define the concepts *limiting element, closed and compact* relative to subclasses $\mathfrak{R}$ of $\mathfrak{D}$.‡ Also if μ is a real-valued function on $\mathfrak{R}$, we can define the notion of *continuity,* as well as *equal continuity,* as applied to a set of functions. In such a situation we are able to state the following

THEOREM. *If $\mathfrak{R}$ is a closed and compact subclass of $\mathfrak{D}$, and μ_{nr} is a monotonic sequence of functions continuous on $\mathfrak{R}$ and*

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* Cf. Dini-Lüroth-Schepp: Theorie der Funktionen, p. 148.

† Cf. *Amer. Journ. of Mathematics,* vol. 34, p. 241.

‡ Cf. Fréchet: "Sur quelques points du calcul fonctionnel," *Rendiconti del Circolo Matematico di Palermo,* vol. 22 (1906), pp. 6, 7, 11.

converging to the continuous function μ_r, *then the convergence is uniform and the functions* μ_{nr} *are equally continuous.*

We prove the uniformity of convergence of μ_{nr}. Since $L_n\mu_{nr} = \mu_r$ for every r of the class $\mathfrak{R}$, we have: for every positive e there exists an n_{er} such that if $n \geqq n_{er}$ we have $|\mu_{nr} - \mu_r| \leqq e$. Suppose that we have selected for each e and r the smallest possible value as n_{er}. Then we wish to show that for each e, n_{er} is bounded on the class $\mathfrak{R}$. Suppose, if possible, this were not so for some particular e. Then for every n, there would exist an r_n, such that $n_{er_n} > n$, i. e., $|\mu_{nr_n} - \mu_{r_n}| > e$. On account of the convergence of μ_{nr} to μ_r for every r, no element can recur infinitely often in the set r_n. Then since the class $\mathfrak{R}$ is compact and closed, there will exist a subsequence $r_{n_m} = r'_m$ of r_n, and an r, such that $L_m r'_m = r$. Consider the double sequence $\mu_{nr'_m}$. It is monotonic non-decreasing in n for every m. Moreover on account of the continuity of μ_{nr} and μ_r, we have $L_m L_n \mu_{nr'_m} = L_n L_m \mu_{nr'_m}$. It therefore fulfils the conditions of our lemma, and it follows that $L_n\mu_{nr'_m}$ converges uniformly; i. e., for every positive e there will exist an n_e, independent of m, such that $|\mu_{nr'_m} - \mu_{r'_m}| \leqq e$. By taking the e as the one presupposed above, and $n > n_m$, we obtain a contradiction.

The equal continuity of the functions μ_{nr} is a direct consequence of their uniformity of convergence and continuity.

To obtain a further generalization we presuppose another general class $\mathfrak{P}$. In $\mathfrak{P}$ we shall suppose that there is defined an order relation between triplets of elements: $B_{p_1p_2p_3}$, comparable to $p_1 \leqq p_2 \leqq p_3$. We shall suppose that there exists in the class also the concept of limit, subject to the condition that, if $L_n p_n = p$, then there exists a subsequence having the same limit, such that $B_{p_{n_m}p_{n_{m+1}}p}$ for every m, or $B_{pp_{n_{m+1}}p_{n_m}}$ for every m. If μ is a real valued function on $\mathfrak{S}$, a subclass of $\mathfrak{P}$, then μ is said to be *monotonic non-decreasing* on $\mathfrak{S}$ if for every triplet s_1, s_2, s_3, of $\mathfrak{S}$ such that $B_{s_1s_2s_3}$ we have $\mu_{s_1} \leqq \mu_{s_2} \leqq \mu_{s_3}$. Finally in the composite class $\mathfrak{P}\mathfrak{Q}$,* we obtain a double limit, viz., $L_{mn}p_m q_n = pq$ is equivalent to $L_m p_m = p$ and $L_n q_n = q$. This enables us to define a continuity of functions on a composite range, similar to that of continuity of functions of two variables, viz., μ is continuous on $\mathfrak{S}\mathfrak{R}$ if $L_{mn}s_m r_n = sr$ implies $L_{mn}\mu_{s_m r_n} = \mu_{sr}$. Then we have the following

* Cf. Moore: Introduction to General Analysis, p. 90.

THEOREM. *If $\mathfrak{S}$ and $\mathfrak{R}$ are closed and compact subclasses of $\mathfrak{P}$ and $\mathfrak{Q}$, respectively, if further μ_{sr} is continuous on $\mathfrak{S}$ for every r and on $\mathfrak{R}$ for every s, and if moreover μ_{sr} is monotonic non-decreasing on $\mathfrak{S}$ for every r, then μ_{sr}, considered as a set of functions on $\mathfrak{R}$, are equally continuous, as well as μ_{sr} considered as a set of functions on $\mathfrak{S}$, and μ_{sr} is continuous on $\mathfrak{S}\mathfrak{R}$.*

The proof that μ_{sr}, considered as a set of functions on $\mathfrak{R}$, are equally continuous is an indirect one. The assumption that μ_{sr} is not equally continuous on $\mathfrak{R}$ is shown to be untenable by a use of the property of limit in terms of B, the monotoneity and continuity of μ_{sr}, and the preceding theorem. The details are easily supplied. The equal continuity of the set μ_{sr} on $\mathfrak{S}$, and the continuity on $\mathfrak{S}\mathfrak{R}$ follow at once from the equal continuity on $\mathfrak{R}$.

By specializing the classes $\mathfrak{P}$ and $\mathfrak{Q}$, we get some interesting theorems in special fields. If we take $\mathfrak{P} = 1, 2, 3, \cdots, \infty$, with $B_{p_1p_2p_3}$ defined as $p_1 \leqq p_2 \leqq p_3$, and $\mathfrak{Q}$ as the interval $0 \leqq x \leqq 1$, and note that equal continuity on $1, 2, 3, \cdots, \infty$ is uniform convergence, we get the Dini theorem stated at the outset. If $\mathfrak{P}$ is the linear interval $0 \leqq x \leqq 1$, and $B_{p_1p_2p_3}$ is the same as $p_1 \leqq p_2 \leqq p_3$, and $\mathfrak{Q}$ is the set $1, 2, 3, \cdots, \infty$, we have:

*If a sequence of monotonic non-decreasing functions continuous on a closed interval converges to a continuous function, the convergence is uniform, and the set of functions are equally continuous.**

If $\mathfrak{P}$ is the linear interval $0 \leqq x \leqq 1$, with $B_{p_1p_2p_3}$ equivalent to $p_1 \leqq p_2 \leqq p_3$ and $\mathfrak{Q}$ is the linear interval $0 \leqq y \leqq 1$, we have:

If $f(x, y)$, defined for $0 \leqq x \leqq 1$, $0 \leqq y \leqq 1$, is continuous in x for every y, and in y for every x, and is also monotonic non-decreasing in x for every y, then $f(x, y)$ is continuous in x and y simultaneously.

* Cf. Buchanan and Hildebrandt: *Annals of Mathematics*, ser. 2, vol. 9, p. 123. It is interesting to observe that this theorem and the Dini theorem are special cases of the same theorem.