ON THE SUMMABILITY OF FOURIER'S SERIES.

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1. LET

$$(1)\qquad A_n^{(k)} = \frac{(k+1)(k+2)\cdots(k+n)}{n!} = \frac{\Gamma(n+k+1)}{\Gamma(k+1)\Gamma(n+1)} \quad (n=1, 2, 3, \cdots), \quad A_0^{(k)}=1,$$

so that

$$(2)\qquad \frac{1}{(1-z)^k} = \sum_{n=0}^{\infty} A_n^{(k-1)} z^n, \quad (|z|<1);$$

then the identity

$$\sum_{n=0}^{\infty} A_n^{(k)} z^n = \frac{1}{(1-z)^{k+1}} = \frac{1}{1-z}\cdot\frac{1}{(1-z)^k} = \sum_{\mu=0}^{\infty} z^\mu \cdot \sum_{\nu=0}^{\infty} A_\nu^{(k-1)} z^\nu$$

gives

$$(3)\qquad A_n^{(k)} = \sum_{\nu=0}^{n} A_\nu^{(k-1)} = \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)}.$$

The nth Cesàro mean of order k of a given series $u_0 + u_1 + \cdots + u_n + \cdots$ is, by definition, equal to

$$(4)\qquad s_n^{(k)} = \frac{1}{A_n^{(k)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(k)} u_\nu = \frac{1}{A_n^{(k)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} \sum_{\mu=0}^{\nu} u_\mu$$

(both definitions being equivalent on account of (3)), and if $\lim_{n=\infty} s_n^{(k)}$ exists and equals s, the given series is said to be summable by Cesàro's means of order k, or briefly, summable (Ck), with the sum s.

In the present note, I propose to give a simplified proof of the following theorem, due to Riesz and Chapman:*

* M. Riesz, "Sur les séries de Dirichlet et les séries entières," *Comptes rendus de l'Académie des Sciences* (Paris), vol. 149 (1909), pp. 909–912 (gives no details of the proof).

S. Chapman, "Non-integral orders of summability of series and integrals," *Proceedings of the London Mathematical Society*, ser. 2, vol. 9 (1911), pp. 369–409. (See p. 390.)

Let $f(x)$ be a function defined in the interval $-\pi \leqq x \leqq \pi$, and such that in this interval $|f(x)|$ is integrable in the Lebesgue sense; then the Fourier series for $f(x)$ is summable (Ck) for any $k > 0$ with the sum $\frac{1}{2}(f(x+0) + f(x-0)) = \frac{1}{2}\lim_{\epsilon=0}(f(x+\epsilon) + f(x-\epsilon))$ at any point where this limit exists.† The convergence of the Cesàro means of order k towards this limit is uniform on any closed range for every point of which $f(x)$ is bounded and $f(x+0) + f(x-0)$ exists uniformly.*

2. To prove this theorem, we start from the well known expression for the sum of the $n+1$ first terms of the Fourier series for $f(x)$.

$$s_n\{f(x)\} = \frac{1}{\pi}\int_{-(\pi+x)/2}^{(\pi-x)/2} f(x+2y)\frac{\sin(2n+1)y}{\sin y}dy$$

$$= \frac{1}{\pi}\int_{-(\pi/2)}^{\pi/2} f(x+2y)\frac{\sin(2n+1)y}{\sin y}dy$$

on account of the periodicity of $f(x)$, as established in footnote *, and this expression is easily transformed into

$$s_n\{f(x)\} = \frac{1}{\pi}\int_0^{\pi/2}(f(x+2y) + f(x-2y))\frac{\sin(2n+1)y}{\sin y}dy,$$

and the nth Cesàro mean of order k of the Fourier series in question becomes, by (4),

$$s_n^{(k)}\{f(x)\} = \frac{1}{\pi}\int_0^{\pi/2}(f(x+2y) + f(x-2y))s_n^{(k)}(y)dy,$$

The simplification in the present proof lies in the method of arriving at the inequality (7), which is obtained by Chapman by a method equivalent to the application of Euler's summation formula to the expression.

The present method is applicable also to the corresponding problem in the expansion of a function of two variables in a series of spherical harmonics; see my forthcoming papers in the *Mathematische Annalen:* "Über die Laplace'sche Reihe" and "Über die Summirbarkeit der Reihen von Laplace und Legendre."

* In Chapman's statement of the theorem, $f(x)$ is only required to be integrable in the Lebesgue sense without being absolutely integrable (both requirements being equivalent only when $f(x)$ is bounded for $-\pi \leqq x \leqq \pi$). In Art. 3 of the present note, it is shown by an example that in this form the theorem is not generally true.

† For $x = \pm\pi$, this limit should be replaced by $\frac{1}{2}f[(-\pi+0) + f(\pi-0)]$, which may be included in the expression above by defining $f(x)$ outside of the interval $-\pi \leqq x \leqq \pi$ as periodic with the period 2π.

where

$$(5) \qquad s_n^{(k)}(y) = \frac{1}{A_n^{(k)}} \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} \frac{\sin(2\nu+1)y}{\sin y}.$$

Making $f(x) = 1$, we obtain

$$1 = s_n^{(k)}\{1\} = \frac{2}{\pi}\int_0^{\pi/2} s_n^{(k)}(y)dy,$$

and consequently

$$s_n^{(k)}\{f(x)\} - \tfrac{1}{2}(f(x+0)+f(x-0))$$

$$(6) \qquad = \frac{1}{\pi}\int_0^{\pi/2}(f(x+2y)+f(x-2y)-f(x+0)$$

$$-f(x-0))s_n^{(k)}(y)dy = \frac{1}{\pi}\int_0^{\epsilon} + \frac{1}{\pi}\int_{\epsilon}^{\pi/2},$$

where $0 < \epsilon < \pi/4$.

We now assume $k < 1$;* the main point in our proof consists in showing that

$$(7) \qquad \frac{1}{\pi}\int_0^{\pi/4} | s_n^{(k)}(y) | dy < c_1 \quad (n = 2, 3, 4, \cdots),$$

where c_1, as well as c_2, c_3, $\cdots$ which will be introduced later, are positive constants independent of n. We decompose our integral as follows:

$$\frac{1}{\pi}\int_0^{\pi/4} | s_n^{(k)}(y) | dy = \frac{1}{\pi}\int_0^{\pi/(2n+1)} + \frac{1}{\pi}\int_{\pi/(2n+1)}^{\pi/4}.$$

As we have

$$\left|\frac{\sin(2\nu+1)y}{\sin y}\right| = \left|1 + 2\sum_{\mu=1}^{\nu}\cos 2\mu y\right| \leqq 2\nu+1 \leqq 2n+1,$$

it follows that

$$| s_n^{(k)}(y) | \leqq \frac{1}{A_n^{(k)}}\cdot(2n+1)\sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} = 2n+1,$$

and consequently

$$(8) \qquad \frac{1}{\pi}\int_0^{\pi/(2n+1)} | s_n^{(k)}(y) | dy < \frac{1}{\pi}\int_0^{\pi/(2n+1)}(2n+1)dy = 1.$$

* A series being (uniformly) summable (Ck) is also (uniformly) summable (Ck') with the same sum when $k' > k$ (see Chapman, l. c.), and it is therefore sufficient to prove our theorem for $k < 1$.

To estimate the second part of our integral, we observe that, for $|z| < 1$,

$$\frac{1}{1 - ze^{2yi}} = \sum_{n=0}^{\infty} z^n e^{2nyi},$$

$$\frac{1}{(1-z)^k(1-ze^{2yi})} = \sum_{n=0}^{\infty} A_n^{(k-1)} z^n \cdot \sum_{n=0}^{\infty} z^n e^{2nyi}$$

$$= \sum_{n=0}^{\infty} z^n \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} e^{2\nu yi},$$

or writing $1/z$ instead of z,

$$\frac{z^{k+1}}{(z-1)^k(z-e^{2yi})} = \sum_{n=0}^{\infty} z^{-n} \sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} e^{2\nu yi},$$

whence, by Cauchy's theorem,

$$\sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} e^{2\nu yi} = \frac{1}{2\pi i}\int_C \frac{z^{n+k}dz}{(z-1)^k(z-e^{2yi})}, \tag{9}$$

the integration being performed in the positive sense over a contour C enclosing the points $z = 1$ and $z = e^{2yi}$, and the determinations of z^k and $(z-1)^k$ being taken so that they are real and positive for z real and > 1. We now deform the contour C into a circuit C_1 consisting of (1) the straight line from $z = 0$ to $z = 1 - \eta$, where $\eta > 0$; (2) the circle $z = 1 + \eta e^{\theta i}$, $-\pi \leqq \theta \leqq \pi$; and (3) the straight line from $z = 1 - \eta$ to $z = 0$, followed by a similar circuit C_2 around $z = e^{2yi}$. As $0 < k < 1$, the integral over (2) tends towards zero with η; on (1) and (3) we have $z^k > 0$, and as $(z-1)^k > 0$ for $z = 1 + \eta$, we have $(z-1)^k = e^{-k\pi i}(1-z)^k$ on (1), but $(z-1)^k = e^{k\pi i}(1-z)^k$ on (3), so that, letting η tend towards zero,

$$\frac{1}{2\pi i}\int_{C_1} = \frac{1}{2\pi i}\int_0^1 \frac{z^{n+k}dz}{e^{-k\pi i}(1-z)^k(z-e^{2yi})}$$

$$+\frac{1}{2\pi i}\int_1^0 \frac{z^{n+k}dz}{e^{k\pi i}(1-z)^k(z-e^{2yi})}$$

$$= \frac{\sin k\pi}{\pi}\int_0^1 \frac{z^{n+k}dz}{(1-z)^k(z-e^{2yi})}.$$

We also have

$$\frac{1}{2\pi i}\int_{C_2} = \frac{e^{2(n+k)yi}}{(e^{2yi}-1)^k} = \frac{e^{(2n+k)yi+(k\pi i/2)}}{(2\sin y)^k},$$

this being the residue of the integrand at $z = e^{2yi}$. Denoting by M the minimum of $|z - e^{2yi}|$ for $0 \leqq z \leqq 1$, so that

$$(10) \qquad M = \begin{cases} \sin 2y & \left(0 < 2y \leqq \frac{\pi}{2}\right), \\ 1 & \left(\frac{\pi}{2} \leqq 2y \leqq \pi\right), \end{cases}$$

we then obtain from (9)

$$\left|\sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)} e^{2\nu yi}\right| < \frac{1}{\pi M}\int_0^1 z^{n+k}(1-z)^{-k}dz + \frac{1}{(2\sin y)^k}$$

$$= \frac{1}{\pi}\frac{\Gamma(1-k)\Gamma(n+k+1)}{\Gamma(n+2)}\cdot\frac{1}{M} + \frac{1}{(2\sin y)^k},$$

and consequently

$$|s_n^{(k)}(y)| = \frac{1}{A_n^{(k)}\sin y}\left|\sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)}\sin(2\nu+1)y\right|$$

$$(11) \qquad \begin{aligned} &\leqq \frac{1}{A_n^{(k)}\sin y}\left|e^{yi}\sum_{\nu=0}^{n} A_{n-\nu}^{(k-1)}e^{2\nu yi}\right| \\ &< \frac{1}{\pi}\frac{\Gamma(1-k)\Gamma(n+k+1)}{\Gamma(n+2)A_n^{(k)}}\cdot\frac{1}{M\sin y} + \frac{2}{A_n^{(k)}}\cdot\frac{1}{(2\sin y)^{k+1}} \\ &= \frac{\Gamma(1-k)\Gamma(1+k)}{\pi}\cdot\frac{1}{n+1}\cdot\frac{1}{M\sin y} + \frac{2}{A_n^{(k)}}\cdot\frac{1}{(2\sin y)^{k+1}}. \end{aligned}$$

By Stirling's formula, it is readily seen from (1) that

$$(12) \qquad \frac{2}{A_n^{(k)}} < \frac{c_2}{(n+1)^k},$$

and from (10), (11) and (12) we obtain, for $0 < y \leqq \pi/4$,

$$\begin{aligned} | s_n^{(k)}(y) | &< \frac{\Gamma(1-k)\Gamma(1+k)}{\pi} \cdot \frac{1}{n+1} \cdot \frac{1}{\sin y \sin 2y} \\ &\qquad + \frac{c_2}{(n+1)^k} \cdot \frac{1}{(2 \sin y)^{k+1}} \\ &\leqq \frac{\Gamma(1-k)\Gamma(1+k)}{\pi} \cdot \frac{1}{n+1} \cdot \frac{1}{\frac{2}{\pi} y \cdot \frac{2}{\pi} \cdot 2y} \\ &\qquad + \frac{c_2}{(n+1)^k} \cdot \frac{1}{\left(2 \cdot \frac{2}{\pi} y\right)^{k+1}} \\ &= \frac{c_3}{n+1} \cdot \frac{1}{y^2} + \frac{c_4}{(n+1)^k} \cdot \frac{1}{y^{k+1}}, \end{aligned}$$

whence

$$\begin{aligned} \frac{1}{\pi} \int_{\pi/(2n+1)}^{\pi/4} | s_n^{(k)}(y) | dy &< \frac{1}{\pi} \int_{\pi/(2n+1)}^{\pi/4} \left(\frac{c_3}{n+1} \cdot \frac{1}{y^2} + \frac{c_4}{(n+1)^k} \cdot \frac{1}{y^{k+1}} \right) dy \\ &< \frac{1}{\pi} \left(\frac{c_3}{n+1} \cdot \frac{2n+1}{\pi} + \frac{c_4}{(n+1)^k} \cdot \frac{1}{k} \cdot \frac{(2n+1)^k}{\pi^k} \right) < c_5, \end{aligned}$$

from which inequality and (8) we immediately deduce (7).

For $0 < \epsilon < \pi/4$, we also obtain from (10), (11) and (12)

$$(13) \qquad | s_n^{(k)}(y) | < \frac{c_6}{(n+1)^k} \cdot \frac{1}{\sin^2 \epsilon} \qquad \left(\epsilon \leqq y \leqq \frac{\pi}{2} \right).$$

In (6), make ϵ so small that, δ being a given positive quantity,

$$(14) \qquad \begin{gathered} | f(x+2y) + f(x-2y) - f(x+0) - f(x-0) | < \frac{\delta}{2c_1} \\ \left(0 \leqq y \leqq \epsilon < \frac{\pi}{4} \right); \end{gathered}$$

then

$$(15) \qquad \begin{aligned} &\left| \frac{1}{\pi} \int_0^{\epsilon} (f(x+2y)+f(x-2y)-f(x+0)-f(x-0)) s_n^{(k)}(y) dy \right| \\ &\quad < \frac{1}{\pi} \int_0^{\epsilon} \frac{\delta}{2c_1} | s_n^{(k)}(y) | dy < \frac{\delta}{2c_1} \cdot \frac{1}{\pi} \int_0^{\pi/4} | s_n^{(k)}(y) | dy = \frac{\delta}{2}. \end{aligned}$$

On account of (13), we also have, bearing in mind the absolute integrability of $f(x)$,

$$\left|\frac{1}{\pi}\int_{\epsilon}^{\pi/2}(f(x+2y)+f(x-2y)-f(x+0)-f(x-0))s_n^{(k)}(y)dy\right|$$
$$< \frac{c_6}{(n+1)^k}\cdot\frac{1}{\sin^2\epsilon}\cdot\frac{1}{\pi}\int_{\epsilon}^{\pi/2}|f(x+2y)+f(x-2y)-f(x+0)$$
$$(16)\qquad -f(x-0)|\,dy < \frac{c_6}{(n+1)^k}\cdot\frac{1}{\pi\sin^2\epsilon}\left[\int_{\epsilon}^{\pi/2}|f(x+2y)|\,dy\right.$$
$$\left.+\int_{\epsilon}^{\pi/2}|f(x-2y)|\,dy+|f(x+0)-f(x-0)|\left(\frac{\pi}{2}-\epsilon\right)\right]$$
$$< \frac{c_7}{(n+1)^k\sin^2\epsilon}.$$

After fixing an ϵ satisfying (14), we determine an $N = N(\epsilon)$ so large that (16) becomes less than $\delta/2$ for $n \geqq N$, and (6), (15) and (16) give

$$|\,s_n^{(k)}\{f(x)\} - \tfrac{1}{2}(f(x+0)+f(x-0))\,| < \delta \quad \text{for} \quad n \geqq N,$$

which proves the first part of the theorem. In regard to the second part, it is sufficient to observe that, the range in question being closed, an ϵ and a c_1 may be determined independent of x so that (14) and (16) hold *uniformly* over the range in question.

3. To show that the theorem is not generally true when $f(x)$ is integrable without being absolutely integrable, consider the function of period 2π defined by

$$f(x) = \frac{d}{dx}\left(x^{\nu}\cos\frac{1}{x}\right) \qquad (0 \leqq x \leqq 2\pi).$$

Riemann* has shown that, for $0 < \nu < \frac{1}{2}$, the nth term in the Fourier series corresponding to this function has the asymptotic expression

$$\left(\frac{1}{2\sqrt{\pi}}\sin\left(2\sqrt{n} - nx + \frac{\pi}{4}\right) + \epsilon_n\right)n^{(1-2\nu)/4}, \quad \lim_{n=\infty}\epsilon_n = 0,$$

* B. Riemann, "Ueber die Darstellbarkeit einer Function durch eine trigonometrische Reihe," Gesammelte Werke, second edition (Leipzig, 1892), pp. 227–265. See pp. 260 et seq.

and, as for the summability (Ck) of the series $u_0 + u_1 + \cdots + u_n + \cdots$ it is necessary that*

$$\lim_{n=\infty} \frac{u_n}{n^k} = 0,$$

it follows that, for any $k < \frac{1}{4}$, we obtain a Fourier series which is not summable (Ck) for any value of x by selecting a ν such that $1 - 2\nu > 4k$. By a suitable modification of Riemann's example, we may construct a Fourier series with the corresponding property for any $k < \frac{1}{2}$; for $1 > k \geqq \frac{1}{2}$, I have not been able to decide whether the theorem is true for all integrable (and not only absolutely integrable) functions or not.†

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NOTE ON PIERPONT'S THEORY OF FUNCTIONS.

IN a review, written some years ago, of Pierpont's Theory of Functions of Real Variables, I made the following incorrect statement with regard to the possibility of reversing the order of differentiation of a function $f(x, y)$:‡

"Under the assumption that f_x' exists on $y = b$, f_y' on $x = a$, and that one of them is approached uniformly, it follows as a corollary to the theorem of Moore mentioned above, that the second derivatives f_{xy}'', f_{yx}'' exist at (a, b) and are equal."

The assumptions should be that f_x' exists on $x = a$, f_y' on $y = b$, and that the derivative for x at $x = a$ of the quotient $f(x, y)/(y - b)$ is approached uniformly for values of y different from b. These are the hypotheses, in different words, which Professor E. H. Moore uses in the Lectures referred to on page 124 of the review, and which I intended to reproduce.

I am indebted for this correction to Mr. G. A. Pfeiffer. In a recent letter to me he cited the example $f = xy(x^2 - y^2)/(x^2 + y^2)$ with the agreement that f shall be zero for $x = y = 0$, which

* S. Chapman, l. c., p. 379.

† For $k \geqq 1$, the theorem holds for any integrable function; see for the case $k = 1$ (the theorem holds a fortiori for $k > 1$) L. Fejér, "Untersuchungen über trigonometrische Reihen," *Math. Annalen*, vol. 58 (1904), pp. 51–69.

‡ BULLETIN, vol. 13 (1906), page 125.