congruence $1^{p-2} + 2^{p-2} + \cdots + \left(\dfrac{p-1}{2}\right)^{p-2} \equiv 0 \pmod{p}$.
Professor Putnam discusses the general congruence of this type,
$1^r + 2^r + \cdots + \left(\dfrac{p-1}{2}\right)^r \equiv a \pmod{p}$, and shows that by
expressing a as a fraction it may be given a value for any fixed
r (less than p) that is independent of p.

5. In a paper in the *Proceedings of the Edinburgh Mathematical Society*, Professor Allardice considered a geometrical
transformation in the plane, $\tan \frac{1}{2}\varphi = k \tan \frac{1}{2}\vartheta$, where ϑ is
the angle formed by the enveloping tangents of a curve with
a given straight line l, the axis of transformation, and φ is
the angle formed by l and t, a system of lines through the
intersection of l and t, which envelop the transformed curve of
c. In the present paper, Dr. Stager considers analytically
a similar transformation in space and applies it to certain
systems of spheres. The method of transformation applied
to space is as follows: Let α be a given plane and P be any
solid. Further, let a plane β be tangent to P and intersect α
in i, making with α an angle ϑ. If through i we draw a plane
β', making with α an angle φ, such that $\tan \frac{1}{2}\varphi = k \tan \frac{1}{2}\vartheta$,
the envelop of β' is defined as the "transform of P." The
paper concludes with a number of applications of the method.

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W. A. MANNING,
Secretary of the Section.

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THE TOTAL VARIATION IN THE ISOPERIMETRIC PROBLEM WITH VARIABLE END POINTS.

BY DR. A. R. CRATHORNE.

IN the simple problem of the calculus of variations,

$$J = \int_{x_1}^{x_2} F(x, y, x', y')dt = \text{minimum},$$

the total variation can be expressed as an integral of which
the integrand is the Weierstrassian E-function. It is the
object of this note to express in a similar way the total vari-

ation for those isoperimetric problems in which the so called Weierstrassian construction is possible.

Let k_1: $x = x_1(u)$, $y = y_1(u)$ and k_2: $x = x_2(v)$, $y = y_2(v)$ be two given curves lying wholly or partly in the region R of the problem but not intersecting in this region. The calculus of variations problem under discussion is

$$J = \int_{k_1}^{k_2} F(x, y, x', y')dt = \min.$$

while

$$K = \int_{k_1}^{k_2} G(x, y, x', y')dt = l.$$

Suppose that C_0: $x = x(t)$, $y = y(t)$ is a minimizing curve intersecting the curves k_1 and k_2 in 1 and 2 (figure). C_0
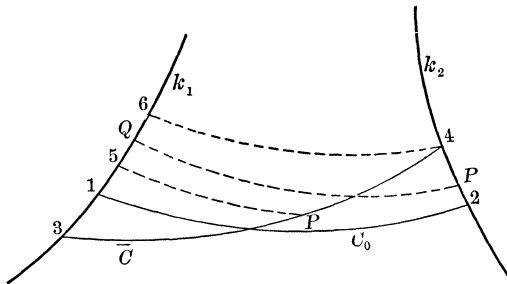
will be one of a two parameter family of extremals which are solutions of the Euler's equation for the problem

$$J = \int H(x, y, x', y')dt = \min., \quad H = F + \lambda G,$$

and which intersect k_1 transversally. Let the equations of this family be

$$x = \varphi(u, \lambda, t), \quad y = \psi(u, \lambda, t),$$

intersecting k_1 for $t = t_1$. From these we have

$$C_0: \quad x = x(t) = \varphi(u_0, \lambda_0, t), \quad y = y(t) = \psi(u_0, \lambda_0, t).$$

Let

$$\chi(u, \lambda, t) = \int_{k_1}^{t} G(x, y, x', y')dt$$

and consider in the usual way the space curves

$$x = \varphi(u, \lambda, t), \quad y = \psi(u, \lambda, t), \quad z = \chi(u, \lambda, t).$$

Since

$$\chi(u, \lambda, t_1) = 0$$

these curves will all intersect k_1 and they can be shown to fill a portion of space and, with the exception of points on k_1, to form a field R'. Let C_0' be the space corresponding to C_0. Suppose $\overline{C}$: $x = \bar{x}(\tau)$, $y = \bar{y}(\tau)$ to be any other curve connecting k_1 with k_2 for which $K = l$. Let $\overline{C}'$ be the corresponding space curve which, since the Weierstrassian construction is possible, will lie in R'. The equations of $\overline{C}'$ are

$$x = \bar{x}(\tau), \quad y = \bar{y}(\tau),$$

$$z = \bar{z}(\tau) = \int_{\tau_1}^{\tau} G(\bar{x}, \bar{y}, \bar{x}', \bar{y}')dt.$$

Now let the point P (figure) move from 3 to 2, taking the path 34 along $\overline{C}$, then 42 along k_2. First considering P as a point on $\overline{C}$, there will be one extremal $5P$ through P cutting k_1 transversely for which $K_{5P} = \overline{K}_{3P}$. We consider in the usual way* $J_{5P} + \overline{J}_{P4}$, and find

$$J_{\overline{C}} - J_{64} = \int_{\tau_3}^{\tau_4} Ed\tau.$$

Following the point P from 4 to 2 along k_2, consider the integral J along the extremals which cut k_1 transversely and for which $K = l$. Let Q be the intersection of these extremals with k_1, then as P moves from 4 to 2, Q will move from 6 to 1. The integral J_{QP} is a function of the parameter v, and using the field integral notation,†

$$J_{QP} = W(x_2(v), y_2(v), z).$$

Since z is constant and equal to l,

$$\frac{dJ_{QP}}{dv} = x_2'W_{x_2} + y_2'W_{y_2} = x_2'H_{x'} + y_2'H_{y'}.$$

The arguments of $H_{x'}$ and $H_{y'}$ are $x_2(v)$, $y_2(v)$, and x' and y'

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* Bolza, Variationsrechnung, p. 507.
† Ibid., p. 504.

for the extremal QP at its intersection with k_2. Let $J_{QP} = S(v)$; then

$$J_{64} = S(v_4), \quad J_{c_0} = J_{12} = S(v_2),$$

$$J_{64} - J_{c_0} = - [S(v_2) - S(v_4)] = - \int_{v_4}^{v_2} \frac{dS}{dv} \, dv$$

$$= - \int_{v_4}^{v_2} [x_2' H_{x'} + y_2' H_{y'}] dv,$$

and we have for the total variation

$$\Delta J = J_{\bar{c}} - J_{c_0} = \int_{\tau_3}^{\tau_4} E d\tau - \int_{v_4}^{v_2} [x_2' H_{x'} + y_2' H_{y'}] dv.$$

At the point 2, C_0 cuts k_2 transversally, that is,

$$x_2'(v) H_{x'} + y_2'(v) H_{y'}|^2 = 0,$$

and in this problem, as in the simple problem of the calculus of variations, we are led to a study of the sign of E along the extremal C_0 when considering sufficient conditions.

UNIVERSITY OF ILLINOIS,
   March, 1913.

A NOTE ON GRAPHICAL INTEGRATION OF A FUNCTION OF A COMPLEX VARIABLE.

BY DR. S. D. KILLAM.

THE object of this paper is to give a shorter and purely graphical method for graphical integration than that of the author in his thesis* on graphical integration of functions of a complex variable.

We can represent a function $f(z)$ of the complex variable $z = re^{i\theta}$ in the $f(z)$-plane by a system of orthogonal curves $r = r_n$ ($n = 0, 1, \cdots, n$) and $\theta = \theta_n$ ($n = 0, 1, \cdots, n$). We choose the values r_n and θ_n so that the $f(z)$ plane is covered by a net of small squares. We seek now a graphical representation in the $Z = X + iY$-plane of the function $Z = \int_0^{z_n} f(z)dz$,

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* " Über graphische Integration von Funktionen einer complexen Variabeln mit speziellen Anwendungen," Dissertation, Göttingen, 1912. Referred to in this paper as " thesis."