$(\mathfrak{M}_L)_* = \mathfrak{M}_*$ (cf. § 44a4). Also if $\mathfrak{M}_L$ has the properties D, Δ, so does $(M_L)_*$ (§ 79.2; § 44a5). In view of these propositions and Theorem II we have the following theorem.

THEOREM III. *If a class $\mathfrak{M}$ is composed of bounded functions μ and has the property D, then the necessary and sufficient condition that $\mathfrak{M}_L$ have the property Δ is that $\mathfrak{M}_*$ have the property Δ.*

In his dissertation, Chicago, 1912, E. W. Chittenden has made very effective use of infinite developments of a range $\mathfrak{P}$ where each stage of the development may contain a denumerably infinite number of subclasses. The theorems here given are valid also for such infinite developments. Theorem I may be established for infinite developments by essentially the same reasoning as above and in fact the same system $((\delta^{ml}))$ used above serves also in the case of infinite developments. The other theorems are established precisely as above.

DARTMOUTH COLLEGE,
February, 1913.

THE ASYMPTOTIC FORM OF THE FUNCTION $\Psi(x)$.

BY MR. K. P. WILLIAMS.

(Read before the American Mathematical Society, April 26, 1913.)

THE function

$$\Psi(x) = -C - \sum_{s=0}^{\infty}\left(\frac{1}{x+s} - \frac{1}{s+1}\right),$$

where C is Euler's constant, is of great importance in many questions in analysis, and also in certain problems in mathematical physics. It is the logarithmic derivative of the gamma function, and plays a fundamental rôle in the study of the latter. On account of the slow convergence of the series which defines it, the knowledge of the asymptotic form of $\Psi(x)$ is particularly desirable.* This can be computed directly from the above expression by the aid of factorial series,†

* We use the term asymptotic according to the definition of Poincaré, and denote such a relation by the symbol $\sim$. See Borel, Les Séries divergentes, p. 26.

† Nielsen, Handbuch der Theorie der Gammafunktion, Kapitel XXI.

or it can be obtained from the definite integral form of the function.* Both of these methods, however, involve a considerable amount of complicated calculation.

We shall show in this paper how to obtain the asymptotic form of $\Psi(x)$ in a very simple manner from its fundamental functional property. As is well known, it is a solution of the non-homogeneous difference equation

$$f(x+1) - f(x) = 1/x, \tag{1}$$

and from this fact we shall derive the asymptotic form of the function, without making use of any, except one very obvious, property of its explicit analytical representation.

1. *The Formal Series.*—We shall merely assume that equation (1) has an analytic solution (all solutions differing from a particular one by a periodic function). It is immediately apparent from the equation itself that this solution increases over a set of points a unit's distance apart on the positive real axis in the manner of the harmonic series. From this we expect our solution to increase like log x.

Let us therefore put

$$f(x) = \log x + a_0 + \frac{a_1}{x} + \frac{a_2}{x^2} + \cdots, \tag{2}$$

and substitute in (1). The quantity $\log(1+x) - \log x = \log(1+1/x)$ can be expanded in a series in $1/x$, which converges for $|x| > 1$. We have also for the same values of x

$$\frac{1}{(1+x)^n} = \frac{1}{x^n}\sum_{s=0}^{\infty} \frac{(-1)^s\binom{n-1+s}{n-1}}{x^s},$$

where, as usual,

$$\binom{m}{r} = \frac{m!}{r!(m-r)!}.$$

The expression which we obtain on substituting can thus easily be written as a series in $1/x$. When we equate the different coefficients to zero we find that a_0 is arbitrary, while

* Nielsen, loc. cit., Kapitel XIV.

$a_1, a_2, \cdots$ are given uniquely by the relations

$$a_1 + \frac{1}{2} = 0,$$

$$-\binom{2}{1} a_2 + a_1 + \frac{1}{3} = 0,$$

$$\binom{3}{2} a_3 - \binom{3}{1} a_2 + a_1 + \frac{1}{4} = 0,$$

$$-\binom{4}{3} a_4 + \binom{4}{2} a_3 - \binom{4}{1} a_2 + a_1 + \frac{1}{5} = 0,$$

$$\cdots\cdots\cdots$$

We shall next show how the a's as determined by the above equations are related in a simple manner to the Bernoulli numbers. Put

$$a_1 = -b_1,$$

$$a_n = \frac{(-1)^{n+1}}{n} b_n \qquad (n > 1).$$

The first of our series of equations then gives $b_1 = 1/2$, while from the $(n-1)$th equation we find

$$b_{n-1} + \frac{1}{2}\binom{n-1}{1} b_{n-2} + \frac{1}{3}\binom{n-1}{2} b_{n-3} + \cdots + \frac{1}{2}\binom{n-1}{1} b_2 - b_1 + \frac{1}{n} = 0.$$

When we add $2b_1 = 1$ to each member, and then multiply by n, this relation takes the form

$$\binom{n}{1} b_{n-1} + \binom{n}{2} b_{n-2} + \cdots + \binom{n}{n-2} b_2 + \binom{n}{n-1} b_1 + 1 = n.$$

Let us now introduce the symbol $\{X + 1\}_n$ to represent the expression obtained on expanding $(X + 1)^n$ by the binomial theorem, and then writing each of the exponents of X as a subscript. With this convention the above recursion formula between the b's takes the very simple form

$$\{b + 1\}_n - b_n = n.$$

But this is precisely one of the symbolic equations which defines the Bernoulli numbers.* It therefore follows that all the b's of odd suffix, except b_1, and accordingly the corresponding a's, are zero.†

In order to make our results agree in notation with those usually given we shall put

$$b_{2n} = (-1)^{n+1}B_n,$$

so that $B_1, B_2, \cdots$ are the constants more commonly called the Bernoulli numbers.

We have thus completely determined all the a's except a_0, which was arbitrary, and to which we shall give the value zero. We then have as a formal solution of (1)

$$f(x) = \log x - \frac{1}{2x} + \sum_{s=0}^{\infty} \frac{(-1)^s B_s}{2s \cdot x^{2s}},$$

where

$$B_1 = \frac{1}{6}, \quad B_2 = \frac{1}{30}, \quad B_3 = \frac{1}{42}, \quad B_4 = \frac{1}{30}, \quad \cdots.$$

2. *The Asymptotic Property of the Formal Series.*—We shall next investigate the relation between the formal solution we have obtained, and an actual solution of (1). In the first place, we see that the formal series diverges for all values of x; for when the subscript p is sufficiently large we have the inequality

$$\frac{B_{p+1}}{B_p} > A(2p+1)(2p+2),$$

where A is a constant.‡ It is therefore natural to study the difference between an actual solution and a certain number of terms of the formal solution.

Let us write

$$\varphi_n(x) = \log x - \frac{1}{2x} - \frac{B_1}{2x^2} + \cdots + (-1)^n \frac{B_n}{2n \cdot x^{2n}},$$

* Cesàro, Elementares Lehrbuch der algebraischen Analysis und der Infinitesimalrechnung. Deutsch von G. Kowalewski, p. 295.

† Cesàro, loc. cit., page 296.

‡ Borel, loc. cit., p. 24.

and then expand $\varphi_n(x+1) - \varphi_n(x)$ in a series in $1/x$. The series so obtained will converge for $|x| > 1$, and from the manner in which the a's in (2) were determined it is evident that the coefficients of $1/x^2, \cdots, 1/x^{2n+1}$ will all be zero,* the remaining coefficients, which we denote by $c_1^{(n)}, c_2^{(n)}, \cdots$, being uniquely determined in terms of $B_1, B_2, \cdots, B_n$. Thus we see that $\varphi_n(x)$ is a solution of the equation

$$(3) \qquad \varphi(x+1) - \varphi(x) = \frac{1}{x} + \frac{c_1^{(n)}}{x^{2n+2}} + \frac{c_2^{(n)}}{x^{2n+3}} + \cdots,$$

where the series converges outside a circle of unit radius.

Suppose next that $g(x)$ is any actual analytic solution of (1) and put

$$\theta_n(x) = \varphi_n(x) - g(x).$$

This new function is then a solution of the equation

$$(4) \qquad \theta(x+1) - \theta(x) = \frac{c_1^{(n)}}{x^{2n+2}} + \frac{c_2^{(n)}}{x^{2n+3}} + \cdots,$$

and consequently is given formally by

$$(5) \quad \theta_n(x) = \omega_n(x) - c_1^{(n)}\left[\frac{1}{x^{2n+2}} + \frac{1}{(x+1)^{2n+2}} + \cdots\right] - c_2^{(n)}\left[\frac{1}{x^{2n+3}} + \frac{1}{(x+1)^{2n+3}} + \cdots\right] \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot \quad \cdot,$$

where $\omega_n(x)$ is some periodic function of period 1.

To show that the double series above converges, we make use of the following inequalities:

$$(6) \qquad \sum_{s=0}^{\infty}\left|\frac{1}{(x+s)^k}\right| < \frac{\pi}{|x|^{k-1}} \qquad (k \geqq 2),$$

if x is in the right half of the complex plane and $|x| > 1$, and

$$(6') \qquad \sum_{s=0}^{\infty}\left|\frac{1}{(x+s)^k}\right| < \frac{2\pi}{|v|^{k-1}} \qquad (k \geqq 2),$$

* In the series from which we determined the a's by equating coefficients to zero, it is found that the coefficient of $1/x^{k+1}$ involves only $a_1, a_2, \cdots, a_k$.

if x is the left half plane* ($x = u + v\sqrt{-1}$) and $|v| > 1$. These inequalities, in connection with the fact that the series in (3) converges, show that the series in (5) represents an analytic function, and consequently the above expression defines $\theta_n(x)$, if $|x|$ is large enough and $\arg x \neq \pm \pi$. We see also that $\omega_n(x)$ must be analytic, since we are assuming the same to be true of $\theta_n(x)$.

We moreover have by (5) and (6) for x in the right half plane

$$|g(x) + \omega_n(x) - \varphi_n(x)| < \pi \left[\frac{|c_1^{(n)}|}{|x|^{2n+1}} + \frac{|c_2^{(n)}|}{|x|^{2n+2}} + \cdots \right].$$

Since the series is absolutely and uniformly convergent in any closed region exterior to the unit circle, this shows at once that

$$\lim_{x=\infty} x^{2n}(g(x) + \omega_n(x) - \varphi_n(x)) = 0,$$

when $-\pi/2 \leqq \arg x \leqq \pi/2$.

It will next be proved that the periodic function $\omega_n(x)$ is in reality independent of n. To show this we merely note that from the above

$$g(x) + \omega_n(x) - \varphi_n(x) \quad \text{and} \quad g(x) + \omega_{n+1}(x) - \varphi_{n+1}(x)$$

both approach zero for x large in the right half plane, while the quantity $\varphi_{n+1}(x) - \varphi_n(x)$ obviously itself approaches zero. It then follows immediately that the difference $\omega_{n+1}(x) - \omega_n(x)$ approaches zero when x goes to infinity along any ray making an acute angle with the positive real axis. But since this quantity is a periodic function, we must therefore have everywhere

$$\omega_{n+1}(x) = \omega_n(x).$$

We can now replace $\omega_n(x)$ by a unique periodic function $\omega(x)$. Then from the above limit and the definition of asymptotic representation we derive the important relation

$$g(x) + \omega(x) \sim \log x - \frac{1}{2x} + \sum_{s=1}^{\infty} \frac{(-1)^s B_s}{2s \cdot x^{2s}},$$

for x in the right half of the complex plane.

* We can obtain these inequalities directly from those derived by elementary means by Birkhoff, "General theory of linear difference equations," *Trans. Amer. Math. Soc.*, vol. 12, p. 248.

If we should make use of (6′) instead of (6) we would find in a similar way that

$$\lim_{v=\infty} v^{2n}(g(x) + \omega(x) - \varphi_n(x)) = 0,$$

for x in the left half plane. But since $v = |x| \sin(\arg x)$, this limit reduces to the former, so that the above asymptotic relation holds for all approaches to infinity, provided only that $\arg x \neq \pm \pi$.

We can now state the theorem: *Given any analytic solution* $g(x)$ *of* (1), *there exists a unique analytic periodic function* $\omega(x)$ *such that* $g(x) + \omega(x)$ *is represented asymptotically by the formal solution of* (1), *for* x *approaching infinity in any direction except along the negative real axis.*

If instead of the value of $\theta_n(x)$ given by (5) we should take the solution of (4) which is analogous to the formal solution

$$l(x - 1) + l(x - 2) + \cdots$$

of the equation

$$h(x + 1) - h(x) = l(x),$$

we could show in a similar way that there exists a periodic function such that its sum and the solution of (1) remains asymptotic to the formal solution for all approaches to infinity except the *positive* real axis.

Let us now take the solution $\Psi(x)$ of (1) and determine the corresponding periodic function. Let $x = n$, a positive integer; then

$$\Psi(n) = -C + 1 + \frac{1}{2} + \cdots + \frac{1}{n-1},$$

whence we see that $\Psi(n) - \log n$ approaches zero when n increases indefinitely. It therefore follows that the periodic function must in this case be zero along the real axis and, being analytic, is consequently identically zero. We thus have the well known relation

$$\Psi(x) \sim \log x - \frac{1}{2x} + \sum_{s=1}^{\infty} \frac{(-1)^s B_s}{2s \cdot x^{2s}},$$

for $-\pi < \arg x < \pi$.

It is evident that the above method and the relation

$\Gamma'(x) / \Gamma(x) = \Psi(x)$ give a very easy way to compute as many terms as may be desired in the series which occurs in the asymptotic form of the gamma function.

INDIANA UNIVERSITY,
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AN ERRONEOUS APPLICATION OF BAYES' THEOREM TO THE SET OF REAL NUMBERS.

BY DR. EDWARD L. DODD.

(Read before the American Mathematical Society, January 1, 1913.)

BAYES' theorem on the probability of causes is frequently introduced with an urn problem.* Here only a finite number of objects come into consideration. For example: The urn U_1 contained 3 white balls and 1 black ball; the urn U_2 contained 2 white balls and 2 black balls. A man, blindfolded, drew a white ball. What is the probability that this white ball came out of U_1,—assuming that each urn was equally accessible? After a consideration of the general problem of this nature, the following theorem, known as Bayes' theorem, is announced:

Let ω_i be the probability a priori that a certain urn, or "cause," or set of conditions U_i will come into play. The "causes" are to be mutually exclusive; and $i = 1, 2, \cdots, s$. Let p_i be the probability that U_i, when brought into play, will yield a certain event. Then, after this event has happened, the probability a posteriori P_i that the event had its origin in U_i is

$$P_i = \frac{\omega_i p_i}{\omega_1 p_1 + \omega_2 p_2 + \cdots + \omega_s p_s}.$$

In the preceding example, it is assumed that $\omega_1 = \omega_2 = \frac{1}{2}$. Hence, with $p_1 = \frac{3}{4}$, $p_2 = \frac{2}{4}$, it follows that $P_1 = \frac{3}{5}$,—a result which on inspection seems plausible; since $\frac{3}{5}$ of all the white balls were in the first urn, U_1. This urn example illustrates, indeed, the following important corollary of Bayes' theorem:

If each of a finite number s of mutually exclusive causes

* E. g., Poincaré, Calcul des Probabilités (1912), p. 153.