mations (1), (2), and (3) be denoted by $\varphi(z_3)$. The latter function is transformed by (4) into $\varphi(-z_3 + 2a)$. From the hypotheses of the theorem and the properties of the transformations employed, it follows that the function

$$\varphi(z_3) \cdot \varphi(-z_3 + 2a)$$

is analytic throughout the interior of σ and vanishes at every point of the boundary. Hence both the real and the pure imaginary parts of this function vanish at every point of the boundary of σ and are, therefore, both identically zero, since a function that is single-valued and harmonic throughout the interior of a region and vanishes at every point of the boundary is identically zero.* Since

$$\varphi(z_3) \cdot \varphi(-z_3 + 2a) \equiv 0,$$

one of the factors vanishes identically, and therefore

$$f(z) \equiv 0.$$

MATHEMATICAL PHYSICS AND INTEGRAL EQUATIONS.

Die Integralgleichungen und ihre Anwendungen in der mathematischen Physik. Vorlesungen an der Universität zu Breslau, gehalten von ADOLF KNESER. Braunschweig, Vieweg, 1911. 8vo. viii+243 pp.

THE solution of various boundary value problems for a partial differential equation by means of the expansion of an arbitrary function in series of solutions of ordinary differential equations involving a parameter constitutes one of the most important applications of the theory of integral equations. Here, as so often elsewhere, mathematical physics has first propounded the question, and it has been the task of analysis to furnish the answer. Especially close, therefore, has been the connection between mathematical physics and integral equations; especially interesting must be likewise a method of treatment which aims to exhibit this connection as vividly as possible. Such is the method of Kneser's book; we learn

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* Osgood, Lehrbuch der Funktionentheorie, vol. 1, 2d ed., 1912, p. 623.

from the preface that the author deals particularly with the applications, and makes the least possible appeal to the general theory. An idea of the fidelity of adherence to this plan may be obtained from the titles of the chapters: integral equations and the linear flow of heat; integral equations and oscillations of linear systems of masses; integral equations and the Sturm-Liouville theory; flow of heat and oscillations in regions of two or three dimensions; existence theorems and the Dirichlet problem; the Fredholm series.

Chapter I studies the temperature state in a straight rod or a ring immersed in a medium of constant temperature. The Green function is not dragged in by the heels for the sake of a possible ultimate utility, but appears naturally as an expression for the temperature state independent of the time (steady flow), produced by the imposition of a heat source of unit strength at a point of the bar or ring. That a source cannot produce steady flow in a rod when the escape of heat from side and ends is prevented, appears physically evident, so that in this case a Green function is impossible; considerations again of purely physical character lead to the usual generalized Green function. Various specializations yield, for the straight rod, the Fourier sine series, the Fourier cosine series, and other trigonometric expansions; for the ring, the complete Fourier series. As regards the general theory, this first chapter already contains the theorem on the expansion of an arbitrary function in terms of the principal solutions for a real symmetric kernel—much of the work, however, being based on the as yet unproved assumption that for such a kernel there is at least one principal parameter value.

In Chapter II we re-discover several formulas of the previous chapter, now clothed with a mechanical instead of a thermodynamic interpretation. The Green function is the displacement produced by a force function which is zero except at a single point. For this concept the author claims neither physical accuracy nor mathematical meaning a limit process may be used to clarify matters. The vibrating string leads us anew to the Fourier sine series. The transverse oscillations of a freely suspended heavy cord bring a result not previously obtained—the expansion in Bessel's functions of order zero; the work is momentarily only formal, as the necessary convergence theorems are postponed to the following chapter.

Indications are given for the similar treatment of a weightless cord rotating about an axis perpendicular to itself, leading to expansions in zonal harmonics (Legendre polynomials). The general theory is enriched by the first appearance of the resolvent function to the kernel of an integral equation.

The general Sturm-Liouville equation arrives in the next chapter, by way of the flow of heat in a non-homogeneous rod; the results of Chapter I are verified and extended. A lacuna is filled, for the cases in hand, by a proof that the kernel corresponding to a Sturm-Liouville equation gives an infinite number of real principal values for the parameter. Bessel's functions and zonal harmonics, which satisfy differential equations whose coefficients are not bounded, and which therefore escape the Sturm-Liouville theory, receive independent treatment. The existence of a unique solution of a linear differential equation of the second order for given values of function and first derivative is proved by the familiar machinery of successive approximations; the real gist of this work is an existence theorem, not (except indirectly) for a differential equation, but for an integral equation of the Volterra type,—the very form of the approximations emphasizes this.* The proof thrown into this form would have been more in keeping with the subject matter of the book, and would have been especially welcome in view of the fact that no other problem discussed by the author leads to a Volterra equation.

A new chapter extends the previous results to the plane and to space. Many interesting problems are solved; the only new point for the theory is the solution of an integral equation with discontinuous kernel by considering instead an equation with properly chosen iterated kernel. The author asserts (in other notation) of the Green function $K(\xi, \eta; x, y)$ for Laplace's equation that

$$K(\xi, \eta; x, y) = -\frac{1}{2\pi}\log\sqrt{(x-\xi)^2+(y-\eta)^2}+M(\xi, \eta; x, y),$$

where M is a function of ξ, η, x, y continuous throughout the region. If this is understood to imply that ξ, η, x, y range independently over the region, the statement is not quite

* Cf. Mason, New Haven Mathematical Colloquium, p. 176.

accurate; E. E. Levi has shown* that although for common approach of (ξ, η) and (x, y) to the same *interior* point the discontinuity of K is completely characterized by the logarithmic term of the preceding formula, for common approach to a *boundary* point the discontinuity is of the nature of twice this logarithmic term.

In the next chapter we find a general proof for the existence of principal solutions for any continuous real symmetric kernel. Explicit formulas are given for the successive principal values; it is interesting to compare here the entirely different expressions obtained for the same purpose by I. Schur.† The author discusses, by a method due to Schmidt, the theorem that solutions of the homogeneous equation with unsymmetric kernel and solutions of the "transposed" equation occur simultaneously; but Kneser's presentation is incomplete. It is shown that solutions for $K(x, y)$ and for another kernel, which we shall call $K_1(x, y)$, do occur simultaneously; as $K_1(x, y)$ is of such form as to render it evident that $K_1(x, y)$ and $K_1(y, x)$ possess or fail to possess solutions simultaneously, the author regards the theorem as proved. The fallacy lies in the fact that $K_1(y, x)$ is not the same as the kernel $K_2(y, x)$ derived from $K(y, x)$ by the corresponding steps to those which evolved $K_1(x, y)$ from $K(x, y)$; Schmidt's own treatment completed the proof by showing that solutions of $K_1(y, x)$ and $K_2(y, x)$ occur simultaneously.

Several further applications to physical problems close this chapter.

At last, in Chapter VI, we come to the direct mathematical treatment of the integral equation. The case in which the parameter does not take a principal value is studied by use of the Fredholm functions $D(\lambda)$, $D(x, y; \lambda)$; the proof given for the Hadamard determinant theorem is elegant in its closeness to the geometric meaning of the theorem. The existence of at least one principal value for a real symmetric kernel is proved again, by a method due to Kneser himself; it is also shown that all poles of the resolvent function are simple. The book closes with a proof that the order of a root of the determinant for any real symmetric kernel having only

* For Green's functions of the second kind; *Göttinger Nachrichten* (1908), p. 248. The fact had been noted, at least for special forms of regions, by earlier writers.

† *Math. Annalen*, vol. 67 (1909), p. 306.

positive principal values (the latter restriction, however, is not essential for the truth of the theorem) is equal to the number of principal solutions corresponding to that root. It is to be regretted that no treatment is given for the case that the parameter takes a principal value.

The contents of the book before us have been described in some detail; what is to be said of it as a whole? That a physicist previously unacquainted with the properties of integral equations will succeed in obtaining any thorough familiarity with them from Kneser's presentation appears very doubtful. First a special problem, then a bit of theory, then more problems, and so on—theoretically this is an admirable plan for teaching or learning a subject; but in the present instance there is seldom a clear line of demarcation between what is always true, what is usually true, and what is true in some special case before us. To the novice the effect would probably be confusing in a high degree. Clearness is also not furthered by the author's building half the theory on the assumption that a symmetric kernel has at least one principal value, and then giving one special and two general proofs of this theorem at so late a stage that careful observation is needed to assure oneself that the vicious circle is avoided. It is the reviewer's belief that a more satisfactory order of presentation would have been obtained by considering first some one simple problem—for instance, one leading to the Fourier sine series; next proceeding to the general Fredholm theory, with the results of Schmidt and of Kneser himself for the symmetric kernel; and then taking up the many other special cases which are discussed.

But to one already familiar with the general theory of integral equations the book is of the highest value. Nowhere else are the details of the application to various physical problems so exhaustively discussed; nowhere else is seen so clearly the physical meaning, not merely of the broad outlines, but of the important separate notions in the theory. Kneser's work furnishes a mine of valuable material for illustrations which illuminate the true import of an integral equation.

That a text containing so many calculative manipulations as this does should have many misprints is to be expected. Some twenty-five have come to the reviewer's notice during a reading none too careful in examination of typographical details. Few are serious; some which might cause difficulty will be noted. On page 78, line 20, for $- H$ read $- H/k(1)$. On page 87, line 12, each denominator Δ should be replaced by $k\Delta$. On page 97, line 8, for $1 + |h'/\rho|$ read $2 + |h'/\rho|$, and make corresponding changes in the succeeding lines. On page 98, line 2, for $- (P' + \epsilon_n)$ read $+ (P' - \epsilon_n)$. On pages 190–197 there is continual confusion of the principal values and their reciprocals.

The general appearance of the page is clear and neat. The functional notation fx instead of $f(x)$ is not at present very widely used, but leads to no confusion here.

WALLIE ABRAHAM HURWITZ.

CORNELL UNIVERSITY,
January 14, 1913.

SHORTER NOTICES.

The Teaching of Mathematics in Secondary Schools. By ARTHUR SCHULZE. New York, The Macmillan Company, 1912. 16 mo. xx+367 pp.

IN these piping times when all readers of fifteen-cent magazines, and other patriots, are hastening to climb on the Progressive band wagon, there is grave offense in describing any person or thing as "conservative"; even the anæmic word "moderate" is eyed askance. We do not wish to create an unfavorable opinion of the book before us by attaching to it any of these unpopular predicates; we prefer to call it "eminently sane." The author is an experienced teacher, the difficulties that he faces are those that actually occur in practice, and the ways that he suggests to meet them are sensible and practical. Perhaps the book may be criticized for being a trifle too practical; a little more might be left to the imagination, there is a superabundance in the wealth of detailed illustration which becomes wearisome to the general reader. This is by design, not inadvertence, as the author shows in the preface (page vi) where, in referring to the books of Smith and Young he says: "This book covers a much more restricted field, but does it in greater detail." Perchance he is right. Surely there are a number of teachers who can obtain a good deal more benefit from a chapter on "The equality of triangles" with one hundred twenty-two illustrative examples, than from a comparison of the heuristic method with the individual mode.