THE FEBRUARY MEETING OF THE AMERICAN MATHEMATICAL SOCIETY.

THE one hundred and sixty-second regular meeting of the Society was held in New York City on Saturday, February 22, 1913. The attendance at the two sessions included the following thirty-eight members:

Professor R. C. Archibald, Mr. H. Bateman, Mr. R. D. Beetle, Mr. A. A. Bennett, Dr. Henry Blumberg, Professor Joseph Bowden, Professor E. W. Brown, Professor B. H. Camp, Professor A. B. Coble, Dr. Emily Coddington, Professor F. N. Cole, Dr. Elizabeth B. Cowley, Miss L. D. Cummings, Mr. C. H. Currier, Dr. L. L. Dines, Professor W. B. Fite, Mr. G. M. Green, Professor C. C. Grove, Professor C. N. Haskins, Mr. S. A. Joffe, Professor Edward Kasner, Dr. N. J. Lennes, Mr. P. H. Linehan, Professor James Maclay, Dr. R. L. Moore, Professor Frank Morley, Mr. F. S. Nowlan, Professor W. F. Osgood, Mrs. Anna J. Pell, Professor R. G. D. Richardson, Professor L. P. Siceloff, Mr. L. L. Smail, Professor D. E. Smith, Dr. M. O. Tripp, Professor Oswald Veblen, Mr. H. E. Webb, Professor H. S. White, Professor W. A. Wilson.

Ex-President H. S. White occupied the chair, being relieved by Professors E. W. Brown and F. Morley. The Council announced the election of the following persons to membership in the Society: Professor E. P. Adams, Princeton University; Dr. H. L. Agard, Williams College; Professor Fiske Allen, Kansas State Normal School; M. Farid Boulad, Egyptian State Railways; Professor J. A. Caparo, Notre Dame University; Mr. C. H. Clevenger, Kansas State Agricultural College; Dr. A. L. Daniels, Jr., Yale University; Mr. W. Van N. Garretson, University of Michigan; Mr. G. M. Green, Columbia University; Mr. C. E. Love, University of Michigan; Dr. Thomas Muir, Education Office, Capetown, S. A.; Mr. J. A. Nyberg, University of Wisconsin; Dean Marion Reilly, Bryn Mawr College; Professor B. L. Remick, Kansas State Agricultural College; Professor W. V. Skiles, Georgia School of Technology; Mr. J. N. Vedder, University of Illinois. Five applications for membership in the Society were received.

The early publication was announced of the Princeton Colloquium Lectures, delivered at Princeton in 1909 by Professors G. A. Bliss and Edward Kasner.

The following papers were read at this meeting:

(1) Professor HARRIS HANCOCK: "A theorem in the analytic geometry of numbers."

(2) Professor B. H. CAMP: "The expression of a multiple integral as a simple integral."

(3) Mr. G. M. GREEN: "Projective differential geometry of triple systems of surfaces."

(4) Dr. C. A. FISCHER: "A generalization of Volterra's derivative of a function of a curve."

(5) Mr. L. B. ROBINSON: "Notes on the theory of systems of partial differential equations."

(6) Professor OSWALD VEBLEN and Mr. J. W. ALEXANDER, II: "Manifolds of n dimensions."

(7) Professor R. G. D. RICHARDSON: "Oscillation theorems for linear homogeneous self-adjoint partial differential equations with one parameter."

(8) Miss L. P. COPELAND: "Concerning the theory of invariants of plane n-lines."

(9) Dr. T. H. GRONWALL: "On the summability of Fourier's series."

(10) Dr. T. H. GRONWALL: "On Lebesgue's constants in the theory of Fourier's series."

(11) Dr. T. H. GRONWALL: "On the degree of convergence of Laplace's series."

(12) Dr. N. J. LENNES: "Note on Lebesgue and Pierpont integrals."

(13) Dr. N. J. LENNES: "Finite sets and the foundations of arithmetic."

(14) Mr. H. BATEMAN: "The expression of the equation of the general quartic curve in the form $A/xx' + B/yy' + C/zz' = 0$."

(15) Mr. H. BATEMAN: "Sonin's polynomials and their relation to other functions."

(16) Dr. DUNHAM JACKSON: "On the accuracy of trigonometric interpolation."

(17) Mr. C. E. WILDER: "On the degree of approximation to discontinuous functions by trigonometric sums."

(18) Professor EDWARD KASNER: "Systems of curves connected with equilong transformations."

Dr. Fischer was introduced by Professor Cole, Mr. Robinson by Dr. Cohen. Miss Copeland's paper was communicated to the Society through Professor Glenn, Mr. Wilder's through

Dr. Jackson. In the absence of the authors the papers of Professor Hancock, Miss Copeland, Dr. Gronwall, Dr. Jackson, and Mr. Wilder were read by title. Abstracts of the papers follow below. The abstracts are numbered to correspond to the titles in the list above.

1. Professor Hancock's paper is in abstract as follows: Let the coordinates of a point in three-dimensional space be denoted by (x, y, z), and further assume that these quantities take only positive integral values. A point is called a unit point if the greatest common divisor of its coordinates is unity. An asymptotic value is found for N, the number of unit points of a fixed rectangular parallelopipedon. It is then evident that the probability that a point is a unit point, when chosen at random among the points (x, y, z), where x varies from k to k_1, y varies from m to m_1 and z varies from n to n_1, is $N/(k_1 - k)(m_1 - m)(n_1 - n)$. In particular, if k, m, n remain fixed, while at least two of the numbers k_1, m_1, n_1 become very large, this probability is $\frac{5}{6}$ nearly.

2. Professor Camp's paper presents some general theorems, of which the following are corollaries:

1. If the function u is defined, limited, and Lebesgue integrable in the multiple, limited field A, and possesses the property that the set of points in A where u remains constant has measure zero, and if v is any other absolutely Lebesgue integrable function defined in A, then the following results hold: $U(x)$ and $V(x)$ exist in the interval $(0, a=\text{meas } A)$ so that, if $B_{U(x)}$ is the set in A where $u < U(x)$, $x = \text{meas } B_{U(x)}$, U is monotone increasing, and

$$\int_0^x U dx = \int_{B_{U(x)}} u dA, \quad \int_0^x V dx = \int_{B_{U(x)}} v dA, \quad \int_0^x UV dx = \int_{B_{U(x)}} uv dA.$$

2. If also $U(x)$ is an integra ,

$$\int_A uv dA = U(a) \int_A v dA - \int_0^a \left(U'(x) \int_{B_{U(x)}} v dA \right) dx.$$

The theorems are useful in extending to multiple integrals theorems that have been established for simple integrals only. In particular this is true in the case of a certain theorem of Lebesgue (*Annales de la Faculté de Toulouse,* series 3, volume 1 (1909), page 65, v).

3. Mr. Green bases a projective theory of triple families of surfaces on the consideration of a certain completely integrable system of six partial differential equations of the second order, of which

$$y^{(k)} = f^{(k)}(u, v, w) \qquad (k = 1, 2, 3, 4) \tag{1}$$

are a fundamental system of solutions. Any fundamental system of solutions will then give a projective transformation of the system (1). The equations (1) give, for $u =$ const., $v =$ const., $w =$ const., the three families of surfaces, if the y's be interpreted as homogeneous coordinates of a point in space. The transformations

$$\begin{gathered} y = \lambda(u, v, w)\bar{y}, \\ \bar{u} = U(u), \qquad \bar{v} = V(v), \qquad \bar{w} = W(w) \end{gathered} \tag{2}$$

are the most general transformations which leave unchanged the triple family of surfaces, without interchanging the families. According to the general method of Professor Wilczynski, the invariants and covariants of the system of differential equations under the transformations (2) are calculated; the geometric interpretation of these invariants and covariants constitutes a projective theory of the triple system.

4. Volterra has defined the derivative of a function of a curve at a point, and has proved that, if it satisfies certain conditions, the first variation of the function $L(C)$ can be expressed by the equation

$$\delta L(C) = \int_{x_1}^{x_2} L'(C, t)\, \delta y(t)\, dt,$$

where $L'(C, t)$ is the derivative of $L(C)$ at the point $x = t$, and where x_1 and x_2 are the end points of the curve C whose equation is $y = y(x)$. In the present paper Dr. Fischer has considered only the class of curves which give fixed end values to a set of m functions determined by m ordinary differential equations of the first order. The definition of the derivative is modified so as to apply to functions defined for this class of curves; the theorem mentioned above is proved in a slightly different form, and its application to the Lagrange problem of the calculus of variations is discussed.

5. Mr. Robinson discusses the systems of partial differential equations named by Riquier regular. An error made by Delassus in his extension of the theorem of Cauchy is corrected. Likewise the author considers the passivity conditions of systems of partial differential equations and shows how some of Riquier's rules can be extended and simplified.

6. The paper of Professor Veblen and Mr. Alexander contains a discussion of n-dimensional manifolds by means of matrices reduced modulo two. Certain theorems are proved which reduce to known results for two-sided manifolds, but which are new for one-sided manifolds. The paper will appear in the June number of the *Annals of Mathematics.*

7. By means of his theory of integral equations Hilbert has proved the existence of functions $u(x, y)$ vanishing on the boundary of a region R and satisfying within that region the most general linear self-adjoint partial differential equation of the second order in two variables and containing one parameter

$$(pu_x)_x + (pu_y)_y + q(x, y)u + \lambda k(x, y)u = 0, \quad (p(x, y) > 0).$$

In the orthogonal case ($k \geqq 0$) there are an infinite number of parameter values $\lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \cdots$ for which such solutions exist; in the polar case (k both signs, $q \leqq 0$) there are two infinite sets $0<\lambda_1 \leqq \lambda_2 \leqq \lambda_3 \leqq \cdots$; $0>\lambda_{-1} \geqq \lambda_{-2} \geqq \lambda_{-3} \geqq \cdots$. Professor Richardson has undertaken the investigation of the nature of the various solutions and shows that to a given integer n there correspond in the orthogonal case at least one, in the polar case at least two solutions $u(x, y)$ of the type sought and such that there are n sub-regions of R on the boundary of which each solution also vanishes. After deriving an existence theorem for the non-orthogonal non-polar equation (k both signs, q positive in at least a portion of R), he shows that in this case there is an integer $n_1 \geqq 0$ such that for $n < n_1$ there is no solution of the type sought, while for $n \geqq n_1$ there are at least two.

This discussion, which holds also for equations in three or more independent variables, will appear in the *Mathematische Annalen.*

8. In the first section of Miss Copeland's paper she establishes necessary and sufficient conditions in order that two

factorable ternary forms f_{3m}, g_{3m}, representing m-lines, may have the property that each form is the sum of constants times the m mth powers of the linear factors of the other. In the case where the simplest full invariant of each m-line vanishes it is shown that the plane pencils have a common vertex and are apolar if they have the above property.

In the second part of the paper the general theory of full invariants is studied. The necessity and sufficiency of the ternary annihilators is established by means of symmetric functions and the solution of linear partial differential equations. This is generalized for n variables. The subject of complete systems is treated by the direct extension of the process for binary forms due to Hilbert.* Complete systems of invariants for a triangle, quadrilateral, and pentagon are obtained, and of covariants for a triangle and quadrilateral. It is shown how the invariants may be expressed rationally in terms of the $2m$ independent coefficients of the m-line's form.

9. Dr. Gronwall gives a simplified proof of the theorem due to Riesz and Chapman, that the Fourier series of an absolutely integrable function $f(x)$ is summable by Cesàro's means of order $k > 0$ with the sum $\frac{1}{2}\lim_{\epsilon=0}(f(x+\epsilon)+f(x-\epsilon))$ at any point where this limit exists, and shows by an example that the theorem is not generally true for a function which is integrable without being absolutely integrable.

10. Dr. Gronwall has shown (*Mathematische Annalen*, volume 72, 1912) that the Lebesgue constants ρ_n increase with n, beginning with $n = 1$; and an independent proof was given by Jackson (*Transactions*, 1912); both proofs depend upon the representation of ρ_n by a definite integral. The present paper gives a proof based directly on Fejér's explicit trigonometrical formula for ρ_n, which is preferable from a systematic point of view.

11. In the present paper, Dr. Gronwall considers the Laplace development in a series of spherical harmonics of a function $f(\theta, \varphi)$ of the geographical coordinates on the unit sphere under the assumption that for any two points θ, φ and θ', φ'

$$|f(\theta, \varphi) - f(\theta', \varphi')| < \omega(\delta),$$

* *Math. Annalen*, vol. 33.

where ω is a given function and δ the distance between the two points. It is shown that the remainder after n terms in Laplace's series for $f(\theta, \varphi)$ is numerically less than a constant multiple of $\omega(1/n)\sqrt{n}$, and that there exist functions $f(\theta, \varphi)$ such that at a given point the remainder, for an infinite number of values of n, is numerically greater than $\Omega(1/n)\sqrt{n}$, where Ω is any function such that $\lim_{\delta=0} \Omega(\delta)/\omega(\delta) = 0$.

In the particular case of the Legendre series for a function $f(x)$, $-1 \leqq x = \cos\theta \leqq 1$, these results apply to the end points $x = \pm 1$ and thus complete the investigations of Dr. Jackson (*Transactions*, 1912) which apply to any interval $-1 + \epsilon \leqq x \leqq 1 - \epsilon$, $(\epsilon > 0)$. As in his theorem $\sqrt{n}$ is replaced by $\log n$, it is seen that there is an essential difference between end points and interior points in Legendre's series (which does not exist, for instance, in the Fourier series).

12. In this paper Dr. Lennes compares the definitions of definite integrals given by Lebesgue and Pierpont. Let c denote the continuum on a certain interval ab, and d any subset of c on this interval. Let $c - d = e$. It is shown that if the function is defined on a measurable set d (in the Lebesgue sense) the two definitions are equivalent. If the set d is not measurable the Lebesgue definition does not apply while that of Pierpont does. However in this case the Pierpont integral does not satisfy one of the fundamental requirements, viz.,

$$\int_d f + \int_e f = \int_c f.$$

By adopting a modification of Pierpont's definition which limits its applicability to the field covered by Lebesgue's, it is possible to make the treatment simpler than Lebesgue's.

13. Dr. Lennes' second paper gives a set of assumptions for point sets and from them derives the usual assumptions for arithmetic.

14. In Mr. Bateman's first paper it is shown that the equation of a general quartic curve can be expressed in the form

$$Ayy'zz' + Bzz'xx' + Cxx'yy' = 0,$$

where $xx'yy'zz' = 0$ is the equation of six straight lines. For a certain type of quartic curve this reduction can be effected in an infinite number of ways.

15. The polynomials introduced by Sonin are discussed in Mr. Bateman's second paper. Various definite integrals and expansions are obtained and the polynomials are used to obtain some elementary solutions of the equation of wave motion.

16. If the values of the function $f(x)$, of period 2π, are known at $2n + 1$ points equally spaced throughout a period, a trigonometric sum of order n at most which takes the same values as $f(x)$ at these points can be constructed by means of formulas analogous to those which define the Fourier series of $f(x)$. The question of the convergence of this trigonometric sum to the value $f(x)$, when the number n is indefinitely increased, has been investigated by Faber (*Mathematische Annalen,* volume 69). Dr. Jackson points out that his theorems, recently communicated to the Society, on the accuracy of trigonometric approximation, in conjunction with a lemma proved by Faber, immediately furnish information as to the rapidity of the convergence of the interpolation formula, the results being similar to those obtained in the case of Fourier's series. For example, if $f(x)$ satisfies a Lipschitz condition, the error does not exceed a constant multiple of $(\log n)/n$. A somewhat less simple formula, still determined by a finite number of values of $f(x)$, is found to give an error not exceeding a quantity of the order of $1/n$, when $f(x)$ satisfies a Lipschitz condition. This formula has a further advantage with reference to the possible effect of errors of observation, if the values of $f(x)$ used are subject to such errors.

17. In this paper Mr. Wilder shows that any function $f(x)$, of period 2π, which has in any finite interval no other discontinuities than a finite number of finite jumps, and in any interval not including one of these points of discontinuity satisfies a Lipschitz condition

$$|f(x_2) - f(x_1)| \leqq \lambda |x_2 - x_1|,$$

is approximately represented, at any point x whose distance from the nearest point of discontinuity is at least as great as δ:

1. By means of a certain trigonometric sum of order n at most, with an error not exceeding

$$\frac{1}{n}\left(c_1\lambda + c_2\frac{\nu}{\delta}\right),$$

where c_1 and c_2 (like $c_3, \cdots, c_6$ below) are absolute constants, and ν is the difference between the upper and lower limits of $f(x)$.

2. By means of Fejér's arithmetic mean of the first $n + 1$ terms ($n > 1$) of the Fourier series of $f(x)$, with an error not exceeding

$$\frac{\log n}{n}\left(c_3\lambda + c_4\frac{\nu}{\delta}\right).$$

3. By means of the first $n + 1$ terms ($n > 1$) of the Fourier series itself, with an error not exceeding

$$\frac{\log n}{n}\left(c_5\lambda + c_6\mu\frac{\nu}{\delta}\right),$$

where μ is the number of discontinuities in any interval of length 2π.

18. The systems of curves studied by Professor Kasner play (roughly) the same role in the geometry of the dual variable $u + jv$ ($j^2 = 0$) as the isothermal systems in the geometry of the ordinary complex variable $x + iy$ ($i^2 = -1$). The analogy is not complete, since the Laplace equation $\psi_{xx} + \psi_{yy} = 0$ is replaced by the simpler equation $\psi_{vv} = 0$, but a list of analogous properties (including new results for the isothermal type) is obtained.

F. N. Cole,
Secretary.

THREE OR MORE RATIONAL CURVES COLLINEARLY RELATED.

BY DR. JOSEPH E. ROWE.

(Read before the American Mathematical Society, December 31, 1912.)

Introduction.

The R^n, or rational plane curve of order n, possesses certain sets* of covariant rational point and line curves which

* J. E. Rowe, "Bicombinants of the rational plane quartic and combinant curves of the rational plane quintic," *Transactions*, vol. 13 (July, 1912), pp. 388–389.