AN ADVANCE IN THEORETICAL MECHANICS.

Théorie des Corps déformables.   Par E. COSSERAT et F. COSSERAT.   Paris, A. Hermann et Fils, 1909.   vi+226 pp.

IN a recent review* of the French edition of the first half of Chwolson's monumental treatise on physics, we purposely and explicitly omitted comment upon two extended notes† of the Cosserats on the foundations of analytical mechanics other than first to state that, owing to the difficulty of the notes themselves and the lack of a general theoretical treatment of mechanics in Chwolson's volumes, these notes seemed out of place, and second to promise that these important contributions of the Cosserats should shortly be reviewed.   The Théorie des Corps déformables is a reprint, with repagination, of the second of these notes, and may be obtained separately by those who may want it without the rest of Chwolson's highly valuable treatise.

The underlying idea of the Cosserats is to base analytical mechanics, in its most widely extended sense, not upon Newton's laws or Hamilton's principle, but upon what they call *euclidean action,* and to define the common concepts of mechanics in terms of this euclidean action; to develop a general theory which shall include as special cases newtonian mechanics and its various derivatives or extensions, of which there are so many, hydrodynamics, elasticity, and electromagnetism, and which shall extend in its generality to an infinity of possible non-newtonian systems of mechanics, of which at least one is now familiar to students of relativity. This ambitious task they have certainly accomplished, and from the favor in which their work seems to be received by such authorities as Appell, it is by no means impossible that Hamilton's principle, which up to the present has contained the most general and unifying theory of mechanics, may rapidly become replaced by the Cosserats' euclidean action.

The fundamental geometric element of their system is not the point, but the point carrying a system of rectangular axes, that is, the trirectangular triedral angle.   It is clear that the

* This BULLETIN, volume 18, pp. 497–508, July, 1912.

† Chwolson, Traité de Physique, volume 1, pp. 236–273, and volume 2, pp. 953–1173.

point alone could not be sufficient; for an elastic filament differs from a geometric curve in the way in which a continuous series of trirectangular triedral angles differs from the locus of the vertices of the angles. Consider a function W of two neighboring positions of the triedral angle, that is, a function of the coordinates of the vertex, of the nine direction cosines of the edges of the angle, and of the first derivatives of these coordinates and direction cosines with respect to the time (in the case of a moving body) or with respect to the arc (in the case of the elastic filament).* Any such function W which has the property of being invariant under the transformations of the euclidean group is called the euclidean action; the quantity Wdt (or Wds_0, where ds_0 is the element of arc of an undeformed elastic filament) is called the euclidean action in the time interval dt (or in the element ds_0 of arc), and is likewise invariant under the euclidean group; the function W may contain explicitly the arc coordinate s_0 or (under appropriate conditions) the time t.

If the theory of the elastic surface is desired we have again to determine a function W of two neighboring positions of the triedral angle (that is, of the position of the vertex of the angle, of the direction cosines of the edges, of their first derivatives with respect to the coordinates ρ_1, ρ_2 of position on the surface, and of ρ_1, ρ_2 themselves), so that W shall be invariant under the euclidean group. The case of the dynamics of an elastic filament is formally identical when one of the coordinates ρ_1, ρ_2 becomes the arc and the other the time. And so on for more complicated cases. The condition that W shall be invariant specializes the form of W in a remarkable manner. In the first place the coordinates of the vertex of the angle cannot occur in W, and in the second place the direction cosines and the various first derivatives cannot occur except as implied in the "velocities" ξ, η, ζ or the "angular velocities" p, q, r associated with the triedral angle.†

The authors then consider the total action between the

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* The analogy between the motion of an elastic medium of k dimensions and the equilibrium of an elastic medium of $k + 1$ dimensions is, or should be, well known to all students of mechanics; it is this analogy which enables the authors to give a uniform treatment to dynamic and static problems of different natures.

† The terms "velocities" and "angular velocities" are included in quotation marks because in the statical problem they are not true velocities and angular velocities but the corresponding derivatives with respect to non-temporal coordinates such as ρ_1, ρ_2.

bounding conditions of the problem, for example

$$A = \int_{t_0}^{t_1} W dt, \quad A = \int_{A_0}^{B_0} W ds_0, \quad A = \int \int W dS_0,$$

and they employ the principle of varying action to define the fundamental quantities such as momentum, force, work, kinetic energy. The whole system of mechanics thus becomes one of nominal definition based upon the function W. The generality, uniformity, and precision of the treatment are striking, and this general statement of the method is intended to emphasize as much as possible these characteristics even at the expense of being vague. We shall now, therefore, discuss a particular application in a little detail for the sake of relieving the vagueness.

In the case of the dynamics of a particle the vertex of the triedral angle is alone significant and the edges must be ignored. The function W is then necessarily a function of the velocity v alone, provided we refer the particle to fixed, not moving, axes. The total action is

$$A = \int_{t_0}^{t_1} W(v) dt.$$

The variation of the action, after an integration by parts becomes

$$\delta A = [F\delta x + G\delta y + H\delta z]_{t_0}^{t_1} - \int_{t_0}^{t_1} (X\delta x + Y\delta y + Z\delta z) dt,$$

where

$$F = \frac{dW}{dv} \frac{1}{v} \frac{dx}{dt}, \quad G = \frac{dW}{dv} \frac{1}{v} \frac{dy}{dt}, \quad H = \frac{dW}{dv} \frac{1}{v} \frac{dz}{dt},$$

$$X = \frac{dF}{dt}, \quad Y = \frac{dG}{dt}, \quad Z = \frac{dH}{dt}.$$

Then F, G, H are by definition the components of momentum and X, Y, Z are by definition the component external forces. In like manner

$$X\delta x + Y\delta y + Z\delta z \quad \text{and} \quad F\delta x + G\delta y + H\delta z$$

are respectively the work done by the external force and the

work done by the momentum.* From these definitions we may prove that, as usual, the impulses of the forces and of their moments are respectively the change of the components of momentum and of moment of momentum.

The work done by the forces becomes at once

$$Xdx + Ydy + Zdz = d\left(v\frac{dW}{dv} - W \right),$$

an exact differential, and if we set

$$E = v\frac{dW}{dv} - W$$

and define E as kinetic energy, we have the familiar theorem as to the equality of the work done and the change of kinetic energy. We may define the mass of the particle in a number of ways dependent on the point of view we desire to adopt. Thus we have several masses:

1°, the quotient of the momentum by the velocity, called the maupertuisian mass, $1/v \cdot dW/dv$,

2°, the quotient of the action by half the square of the velocity, the hamiltonian mass, $2W/v^2$,

3°, the quotient of the kinetic energy by half the square of the velocity, the kinetic mass, $2E/v^2$.

Now if it be supposed that the mass is finite and the $W(v)$ is developable in powers of v, the development must begin with the second power,

$$W = \frac{m}{2}v^2 + (\ )v^3 + \cdots.$$

For sufficiently small values of v the terms of order higher than $\frac{1}{2}mv^2$ may be neglected. The system of dynamics founded upon the approximation $W = \frac{1}{2}mv^2$ is none other than the newtonian, and the three masses above defined reduce to the same constant m. Thus newtonian mechanics appears as the theory of slow motions, of motions infinitely near to the state of rest. The distinction between slow and fast motions is indeed analogous to that between infinitesimal and finite displacements in the theory of elasticity. From

* The authors use the term work in both cases despite the evident difference of physical dimensions, just as they use action for W or Wdt. This seems unfortunate.

the point of view of the principle of relativity, motion cannot be infinitely fast; those interested in these theories may discuss the mechanics founded on the assumption $W = m_0(1 - \sqrt{1 - v^2})$.

It would be useless here to follow in detail the three great sections of the text, namely, equilibrium of an elastic filament, equilibrium of an elastic surface and dynamics of an elastic filament, and equilibrium and motion of a continuous medium; they will be followed in detail by all earnest students of these topics. We prefer here to point out an advantage and a disadvantage of the Cosserats' system. In the main these are those associated with the transfer of any deductive-intuitional physical science to the corresponding formal-deductive mathematical discipline. The gain is in sureness, in freedom from constant doubtful appeal to intuition; a great variety of possible assumptions and corresponding cases may be discussed systematically and accurately from a given uniform point of view. Those who have taught the theory of elasticity will most appreciate this advantage. The loss is in the lessened training of that physical intuition, which is vital for the future success of the young physicist and which can be acquired only by practice in making such various plausible, but not demonstrated, assumptions as are frequent in the theory of elasticity. The simplest explanation of the world already subdued may in the last analysis be mathematically formal; but the subjugation of the regions not yet reached can in the first instance be accomplished only by the imagination.

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SHORTER NOTICES.

Opere Matematiche. By GIULIO CARLO DEI TOSCHI DI FAGNANO. Pubblicate sotto gli Auspici della Società Italiana per il Progresso delle Scienze. Per cura dei Professori Senatore Vito Volterra, Gino Loria, e Donisio Gambioli. Rome, 1912. 3 Vols. 40 lire.

IF we were to select from the great mathematicians of the world a half dozen whose works we might deem worthy of