

3. Theorem 1 of § 2 concerned the *consistence* of the relations

$$(6) \quad \phi\{x, \phi(y, z)\} = \phi\{y, \phi(x, z)\},$$

$$(6') \quad \phi\{f(y, z), z\} = y,$$

$$(6'') \quad f\{\phi(y, z), y\} = z.$$

These relations, on the other hand, may be shown to be *completely independent*.* To do this, it suffices to exhibit eight systems, each of which satisfies or contradicts the preceding relations. Denoting by the symbol (+ + -) that (6) is satisfied, (6') is satisfied, (6'') is contradicted, and so on, we have as the required systems:

(+ + +)	$\phi(x, y) = x + y$	$f(x, y) = x - y$
(+ + -)	$= -x + y$	$= -x + y$
(- + +)	$= -x - y$	$= -x - y$
(+ - +)	$= -x + y$	$= x + y$
(+ - -)	$= x + y$	$= x + y$
(- + -)	$= x - y$	$= x + y$
(- - +)	$= x - y$	$= -x + y$
(- - -)	$= x - y$	$= x - y$

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DOUBLE CURVES OF SURFACES PROJECTED FROM SPACE OF FOUR DIMENSIONS.

BY DR. S. LEFSCHETZ.

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1. IN the study of a surface F in S_4 , one of the problems is the determination of the genus π' of the double curve of a projection of F . Severi† has shown that a general F_m in S_4 has four fundamental projective characters, viz., the order m , the class n , the rank a of a hyperplane section, and the number t of trisecants through an arbitrary point O . He also gives an expression for the rank of the double curve in question, from which, knowing the number of pinch points J of F'_m in projection of F_m , we can obtain π' . It is not uninteresting

* Cf. E. H. Moore, The New Haven Mathematical Colloquium (1910), p. 82, §47.

† "Intorno ai punti doppi impropri . . ." *Palermo Rendiconti*, vol. 15 (1901), p. 32.

to show how π' can be obtained directly, and this is what we will do, discussing afterwards the case of a complete intersection in S_4 .

2. If h is the order of the double curve, we have

$$h = \frac{1}{2}m(n-1) - a,$$

$$p_n = \binom{m-1}{3} - h(m-4) + \pi' + 2t - 1,$$

where p_n is the arithmetic genus of F_m . This gives

$$\pi' = h(m-4) + p_n - 2t - \binom{m-1}{3} + 1$$

$$(1) = \frac{1}{6}(n-1)[2n^2 - 7n - 6] - \frac{1}{2}a(n-4) - 2t + p_n + 1.$$

Another method of obtaining π' could be used if we knew the genus π of the curve of which the double curve is the projection. For the two are in (2, 1) relation, and the coincidences occur on the curve in S_4 only and are: (1) 2 for each triple point, or $2t$; (2) one for each pinch point, or j , so that we have*

$$2t + j = 2(\pi - 1) + 4(\pi' - 1),$$

$$(2) \therefore \pi' = \frac{1}{4}[2(\pi - t + 1) - j].$$

3. If we wish to apply either method to the case of an $F_{\mu\nu}$, complete intersection of a V_μ and a V_ν , we must calculate t , and for the first method we must find p_n . The other characters are indeed easily obtained.

$$(3) \quad m = \mu\nu,$$

$$(4) \quad h = \frac{1}{2}\mu\nu(\mu-1)(\nu-1),$$

$$(5) \quad \pi = \binom{h}{2} - 2h[2h - \mu - \nu - (\mu-1)(\nu-1) + 2],$$

$$(6) \quad j = \mu\nu(\mu-1)(\nu-1) = 2h.$$

We obtain (5) by remarking that the curve in S_4 is the complete intersection of V_μ , V_ν and a $V_{(\mu-1)(\nu-1)}$ and applying Veronese's† formula for the number of apparent double points of a curve in n -space, while (6) is obtained by remarking that j is also the number of tangents drawn from O to $F_{\mu\nu}$, and is therefore the number of points common to V_μ , V_ν and their first polars with respect to O .

* Clebsch, *Leçons de Géométrie*, vol. II, p. 169.

† "Behandlung der projectivischen Verhältnisse . . ." *Math. Ann.*, vol. 19.

4. *Value of p_n .* It is easy to show that $F_{\mu\nu}$ is a regular surface. In fact the system $|G'|$ adjoint to the system $|G|$ of the hyperplane sections of $F_{\mu\nu}$ is cut out by all the $V_{\mu+\nu-4}$ which do not pass through F , and they are cut by the hyperplane H of C in the system of all surfaces of order $\mu + \nu - 4$ which do not contain C , surfaces which cut out on C the complete canonic series on the curve.* $|C'|$ cuts therefore on C a series of defect 0. $\therefore p_g - p_n = 0$,† which was to be proved. To find p_g we must obtain the number of conditions that a $V_{\mu+\nu-5}$ shall go through $F_{\mu\nu}$, since the canonic system of F is cut out by the $V_{\mu+\nu-5}$ that do not go through $F_{\mu\nu}$. More generally we will propose to find the number N_i of conditions that a V_i shall go through $F_{\mu\nu}$. For this purpose we consider the curves C_i obtained in cutting by hyperplanes H_i ($i = 1, 2, \dots, l$). C_i is the complete intersection of two surfaces of order μ, ν in H_i , and has for genus

$$p = \frac{1}{2}\mu\nu(\mu + \nu - 4) + 1.$$

Let r_i be the number of conditions that a V_i going through C_1, C_2, \dots, C_{i-1} has to satisfy in order to go also through C_i . Such a variety already meets C_i in $(i-1)\mu\nu$ points, contained in $i-1$ planes. The series $g_{(i-1)\mu\nu}^{i-1}$ which its variable intersections with C_i cut on the curve is complete, since it is also cut out by a complete system of adjoints, namely the surfaces of order $l-i+1 = \alpha$ of H_i , which go through the fixed group of $(i-1)\mu\nu$ points. But the surfaces of order α in H_i , cut out on C_i a series of same degree, complete and contained in the preceding, therefore coincident with it. Hence if $\nu < \mu$ and

$$\alpha < \nu, \text{ then } r_i = \binom{\alpha+3}{3},$$

$$\nu \leq \alpha < \mu, \text{ then } r_i = \binom{\alpha+3}{3} - \binom{\alpha-\nu+3}{3},$$

$$\mu \leq \alpha \leq \mu + \nu - 4, \text{ then } \mu + \nu - 4 = \alpha \leq \nu - 4,$$

and if $r_i - 1$ is the residual special series, we have by applying Riemann-Roch's theorem

$$2(r_i - r'_i) = 2\alpha\mu\nu - 2(p - 1);$$

$$\therefore r_i = \alpha\mu\nu - (p - 1) + \binom{\mu+\nu-\alpha-1}{3},$$

* Noether, "Zur Theorie des eindeutigen Entsprechens . . .," *Math Ann.*, vol. 8 (1874).

† E. Picard et Simart, *Traité des fonctions algébriques de deux variables*, 2, pp. 437, 489.

$$(4) \quad \mu + \nu - 4 < \alpha.$$

Then the series cut out by V_l is not special.

$$\therefore r_i = \alpha\mu\nu - p + 1.$$

It follows that

$$N_l = \sum_1^l r_i + 1.$$

In particular

$$N_{\mu+\nu-5} = \sum_{\alpha=\mu-\nu}^{\mu-1} \binom{p+3}{3} + \sum_{\alpha=\mu}^{\mu+\nu-5} (\alpha\mu\nu - p + 1) + \sum_{\alpha=1}^{\nu-4} \binom{\nu-\alpha}{3}.$$

Also

$$(7) \quad p_g = p_n = N_{\mu+\nu-5};$$

hence (1°) If $l \geq \mu + \nu - 5 \geq \mu \geq \nu$,

$$\begin{aligned} N_l &= \sum_{\alpha=\mu+\nu-4}^l (\alpha\mu\nu - p + 1) + 1 + (p_n - 1) + 1 \\ &= \sum_{\alpha=1}^l (\alpha\mu\nu - p + 1) + p_n + 1 = \mu\nu \binom{l+1}{2} - l(p-1) + p_n + 1 \end{aligned}$$

in virtue of the expression of p .*

(2°) If $l \leq \mu + \nu - 5$,

$$N_l = p_n - \sum_{i+1}^{\mu+\nu-5} (\alpha\mu\nu - p + 1) - \sum_{i=\mu+2}^{\nu-4} \binom{\nu-\alpha}{3}.$$

5. *Value of t .* Let $t = \varphi(\mu, \nu)$. Applying to a V_ν formed of a V_1 and a $V_{\nu-1}$, we have

$$\varphi(\mu, \nu) = \varphi(\mu, \nu-1) + \mu \binom{\nu}{2} (\nu-1)(\nu-2) - \mu(\mu-1)(\nu-1)(\nu-2),$$

where the second term is the number of bisecants of the surface $(V_\mu, V_{\nu-1})$ meeting also the surface (V_μ, V_1) , while the third is the number of these chords going through the intersection of $V_{\nu-1}$, V_μ and V_1 . Taking $\sum_1^5 \varphi(\mu, i)$ and simplifying, we obtain

$$t = \varphi(\mu, \nu) = 6 \binom{\mu}{3} \binom{\nu}{3}.$$

We see that (2) is easiest to apply in the case of a complete intersection, and indeed in this case comparing both values of π' , we have the easiest way of obtaining p_n .

* Cf. Severi, "Su alcune questioni di postulazione," *Palermo Rendiconti*, vol. 17 (1903), p. 87.

6. The curve of S_4 of which the double curve of $F'_{\mu\nu}$ is the projection is a complete intersection, but such is not the case for the double curve itself. In this respect the case where $\mu = \nu = 3$ is instructive. For then it is found that $t = 6\binom{3}{3} = 6$, $p_g = p_n = 5$,

$$\begin{aligned}\pi &= 5 \cdot 18 + 5 - \binom{8}{3} - 2 \cdot 6 + 1 = 28, \\ h &= \frac{1}{2} 3 \cdot 3 \cdot 2 \cdot 2 = 18 = 6 \cdot 3 = 9 \cdot 2.\end{aligned}$$

The complete intersection of an F_9 and an F_2 with six triple points is of genus 63, while the intersection of an F_6 and an F_3 with six triple points is of genus 28, and yet the curve with the above characteristics cannot lie on an F_3 , for then it would meet F_9' in a curve of order 36. If $F_{\nu(\mu-1)(\nu-1)}$ is the intersection of V_ν and $V_{(\mu-1)(\nu-1)}$, then the curve in S_4 is on $F_{\nu(\mu-1)(\nu-1)}$, the residual intersection being of order $\mu\nu(\mu-1)(\nu-1)^2$, and its genus p_1 is found by remarking that if we project from another center this curve will be residual of one of order $\mu\nu(\mu-1)(\nu-1)$ and genus π , so that

$$\pi_1' = \pi + \frac{1}{2} \mu\nu(\mu-1)(\nu-1)(\nu-2) [\mu\nu + \nu(\mu-1)(\nu-1) - 4].$$

Thus, above, the residual curve is of order 72 and genus 397.

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GEOMETRICAL OPTICS.

The Principles and Methods of Geometrical Optics, especially as Applied to the Theory of Optical Instruments. By JAMES P. C. SOUTHALL. New York, The Macmillan Company, 1910. xxiii+626 pp. with 170 figures.

THAT mathematicians and physicists have left the field of geometrical optics so largely to the scientific experts of the best firms of optical engineers may be but one of the signs of our ever increasing specialization and its accompanying narrowing of interests. Yet the association of such names as Euler, Fermat, Gauss, Hamilton, Kummer, Moebius, Sturm shows that once mathematicians contributed largely to the subject and were inspired by it; a similar state of affairs is true in regard to physicists.

From an impartial viewpoint it can hardly be gainsaid that there are at present more points of contact between geometrical