not satisfy the previous conditions. The positive root of the quadratic factor is

$$x = 1.9266.$$

From this

$$q \equiv d = 1.64676$$

and therefore

$$\theta = 31° \, 16'.$$

Hence *the concave arch of the curve* (22) *furnishes a minimum for the integral J beyond the point where the tangent makes an angle of* 31° 16′ *with the positive x-axis.*

A statement similar to those in the two preceding sections regarding the most general solution and the determination of the constants holds for this problem.

It should be noted that in neither of the three cases considered is the angle θ corresponding to $q = d$ as large as that of the newtonian problem where $\theta = 45°$. So whether or not the newtonian law fails for small values of the angle α, it is certain that these laws hold *only* for smaller values of the angle than in the newtonian problem.

SHEFFIELD SCIENTIFIC SCHOOL,
YALE UNIVERSITY.

SHORTER NOTICES.

Les Systèmes d'Équations aux Dérivées partielles. By CHARLES RIQUIER. Paris, Gauthier-Villars, 1910. xxvii + 590 pp.

DURING the past twenty years Professor Riquier has published a large number of memoirs on the theory of systems of partial differential equations. The main results of his investigations are now made more accessible to mathematicians by incorporating them in a systematic treatise where they are presented from a uniform point of view. The theory of the most general system, containing any number of equations involving any number of functions of any number of independent variables with their partial derivatives of arbitrary order—is naturally extremely difficult, and the author is to be congratulated for the clearness of his treatment. The symbolism and terminology are carefully chosen, the main theorems are stated in italics, and the proofs, which are sometimes very long—that for the convergence of the integrals of an orthonome system, for example, occupies thirty pages—are divided into numbered parts, so that the reader may catch his breath.

Five of the fourteen chapters into which the work is divided are of general introductory nature, dealing with continuity, series in general and power series, olotrope functions, analytic prolongation, implicit functions. The treatment is confined to ideas and results used in the sequel. These chapters really constitute a fairly complete treatment of the theory of analytic functions. The author follows Méray in general —Weierstrass is not mentioned—but finds it convenient to modify Méray's definition of an olotrope function by requiring that the region in which the function is defined shall be normal (that is, any two points can be connected and every point is interior). The chapter on implicit functions is very elaborate, covering almost fifty pages. From the outset the author considers functions of n variables, which may be real or complex.

The discussion of analytic prolongation (calcul par cheminement) is rather meager. The difficult subject of the prolongation of the solutions of partial differential equations is treated in a few of the author's memoirs, but as the results are incomplete this aspect of the theory is not included in the present treatise.

In order to economize space the author finds it necessary to introduce a large number of new technical terms. For example, a set of n integers $a_1, a_2, \cdots, a_n$ is said to be of taxe inferieure to another set $b_1, b_2, \cdots, b_n$ provided the first nonvanishing difference $a_1 - b_1, a_2 - b_2, \cdots, a_n - b_n$ is negative. Again, if two systems of analytic equations (numerical or differential) are such that each equation of the one system can be obtained by adding multiples of the equations of the other system (the factors being arbitrary analytic functions), the two systems are said to be in correlation multiplicatoire. An index of these terms would have been of distinct value.

The theory of partial differential equations here presented deals exclusively with power series. Its object is to complete the discussion of the Cauchy-Kowalewski existence theorems. The boundary value problem is thus excluded. The first step is to show that for given initial conditions series can be calculated which formally satisfy the given differential equations; the second step is to show that the series are convergent. This is carried out for orthonome systems (including the Kowalewski type as a very special case) in Chapter VII; the results are extended and simplified in various directions in the following chapters. Besides the general theory the author includes two important applications illustrating the advantage of his methods: the finite deformations of a continuous medium, and the determination of orthogonal curvilinear coordinates, both for n dimensions.

In the final chapter it is shown that, given any system of equations, it is possible by a finite number of steps to find out whether the system is incompatible or compatible; and in the latter event to reduce it to an equivalent completely integrable system whose general solution involves a finite number of arbitraries (functions or constants). The operations involved in the reduction are those that Lie describes as performable, namely, differentiations and eliminations. There are certain unsettled difficulties as to the implicit functions which arise in the reduction. The author states (page 557): "la très grande généralité du problème posé ne nous permettra d'obtenir, dans les raisonnements qui vont suivre, ni une rigueur absolue, ni une précision irreprochable."

In discussing the degree of generality or the degree of infinitude of the solution of partial differential equations, the reviewer has found the following notation very convenient. Let the symbol f_n denote an arbitrary (analytic) function of n independent arguments; in particular f_0 denotes an arbitrary constant. Then the number of particular solutions contained in a general solution involving k_i arbitrary functions of n_i arguments may be denoted by

$$\infty^{\Sigma k_i f_{n_i}}.$$

For example, the number of curves in a plane is ∞^{f_1}; the number of curves in space is ∞^{2f_1}; the number of surfaces in space is ∞^{f_2}; the number of surfaces of revolution is $\infty^{f_1+4f_0}$, that is, ∞^{f_1+4}, since ∞^{f_0} is the same as ∞^1. The notation is capable of abuse and its chief value is heuristic.

Certain fundamental questions of dimensionality suggest themselves. Thus, it is known that the manifolds ∞^n and ∞^m, $n > m$, are distinct in the sense that no continuous one-to-one correspondence can be established between them. This is in all probability true for the manifolds ∞^{nf_i}, ∞^{mf_i}, $n > m$; and for ∞^{f_i}, ∞^{f_j}, $i > j$; but where is a proof to be found? The question is of interest both for analytic functions and for general continuous functions.

EDWARD KASNER.

Vorlesungen über ausgewählte Gegenstände der Geometrie. Erstes Heft: *Ebene analytische Kurven und zu ihnen gehörige Abbildungen.* Von E. STUDY. Leipzig, Teubner, 1911. 126 pp. M. 4. 80.

As the title indicates, this monograph contains the first part of a series of lectures on a number of selected geometric topics and deals with the geometry in the complex domain, defined as a cartesian plane with ordinary complex numbers as coordinates. Points of such a domain are, for example, imaginary points of intersection of algebraic curves.

In the introduction we find a summary of von Staudt's famous treatment of imaginary elements by elliptic involutions. The disadvantage of this method is that in order to apply it to the solutions of even some very simple problems concerning positional relations a very clumsy apparatus has to be set in motion.

It is therefore desirable to devise a scheme by which problems involving imaginaries can be handled in a simpler and more effective manner. This is exactly what Study accomplishes in a very thorough manner in his valuable monograph.

He starts out by defining the first and second picture (erstes und zweites Bild) of an imaginary point (ξ, η) in a plane. Designating by $\bar{\xi}$ and $\bar{\eta}$ the conjugates of ξ and η, the first picture is obtained as the real pair of points of intersections of the left and right handed minimal lines through (ξ, η) and $(\bar{\xi}, \bar{\eta})$. As indicated in a footnote on page 10, this representation is (apparently) due to Laguerre. It is well to state at this point explicitly that as a pioneer in the field of the complex domain Laguerre accomplished as much as any other man and may be justly put on the same level with von Staudt. As early as 1853 Laguerre found the remarkable result that the radian measure of an angle may be defined as the product of $\frac{1}{2}\sqrt{-1} = \frac{1}{2}i$ and the cross-ratio formed by the two sides of the angle and the isotropic lines through its vertex. In two further articles "Sur l'emploi des imaginaires en géométrie,"